1.4.8) \( A^k = \begin{bmatrix} \cos(k\theta) & \sin(k\theta) \\ -\sin(k\theta) & \cos(k\theta) \end{bmatrix} \). This holds because if \( A \) rotates every vector counterclockwise by an angle of \( \theta \), then applying \( A \) \( k \) times means that in total, you rotate it by an angle of \( k\theta \). (This isn’t really a solution to the problem as stated, but it does explain what is going on.)

1.4.14) Let \( A = [a_{ij}] \). The \((i,j)\) entry of \((r + s)A\) is \( ra_{ij} + sa_{ij} = (r + s)a_{ij} \). Because \((r + s)A\) and \( rA + sA \) have the same \((i,j)\) entry for all \( i \) and \( j \), the matrices are equal.

1.4.25) \( sx = A^2x = A(Ax) = A(rx) = r(Ax) = r^2x \), so \( s = r^2 \).

1.5.3) We compute the \((i,j)\) entry of each. Let \( A = [a_{ij}] \) and \( B = [b_{ij}] \). The \((i,j)\) entry of \( AB \) is \( \sum_{k=1}^n a_{ik}b_{kj} \). Because \( A \) and \( B \) are both diagonal, each summand is zero unless \( i = k = j \). Thus, \( AB \) is also diagonal, and its \((i,i)\) entry is \( a_{ii}b_{ii} \).

Similarly, the \((i,j)\) entry of \( BA \) is \( \sum_{k=1}^n b_{ik}a_{kj} \). Again, this is zero unless \( i = k = j \), so \( BA \) is diagonal, and its \((i,i)\) entry is \( b_{ii}a_{ii} \). Thus, every entry of \( BA \) is the same as that of \( AB \).

1.5.10) If \( p = 0 \) or \( p = 1 \), the statement is trivial. Otherwise, suppose that \((cA)^{p-1} = c^{p-1}A^{p-1}\). By Theorem 1.3 (d),
\[
(cA)^p = (cA)(cA)^{p-1} = (cA)c^{p-1}A^{p-1} = ce^{p-1}AA^{p-1} = c^pA^p.
\]
This shows that if the theorem is true for \( p - 1 \), then it is true for \( p \). That is, if it is true when \( p = 1 \), then it is true when \( p = 2 \). Since we know it is true when \( p = 1 \), it must be true when \( p = 2 \). Next, if it is true for \( p = 2 \), then it is also true for \( p = 3 \). Since we now know that it is true for \( p = 2 \), it must be true for \( p = 3 \), and so forth.

This sort of argument is called mathematical induction. I didn’t expect you to produce a proof by induction, though if you did, then good for you. You were really only expected to say something about how Theorem 1.3 (d) lets you pull constants out.

1.5.14) a) If \( A = [a_{ij}] \) is scalar, then it is diagonal, so if \( i \neq i \), then \( a_{ij} = 0 = a_{ji} \). If \( i = j \), then \( a_{ij} \) is the same entry as \( a_{jj} \). As \( a_{ij} = a_{ji} \) for all \( i \) and \( j \), this means that the matrix is symmetric.

b) No, the zero matrix is scalar, but it is singular, as \( AO = O \) always, and so \( AO \neq I \), regardless of \( A \).

c) No. Consider \( \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \). 

1
1.5.17) Let \( A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \). We can compute
\[
AA^T = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.
\]
Meanwhile,
\[
A^T A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.
\]
Thus, \( AA^T \neq A^T A \). Most possible choices of \( A \) would have worked to prove that the two sides aren’t always equal.

1.5.20) If the matrix is scalar, then all entries off of the diagonal are zero. If it is skew-symmetric, then for every \( i \), \( a_{ii} = -a_{ii} \), so \( a_{ii} = 0 \), and all entries on the diagonal are also zero. If all entries are zero, then we have the zero matrix \( O \).

1.6.16) a) \( A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} \), so this swaps the \( x \) and \( y \) coordinates. Hence, it is reflection about the line \( x = y \).

b) \( A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix} \), so this is the same as part (a), except that it then multiplies each vector by -1. If you try some examples, you might see that this is reflection about the line \( y = -x \).

1.6.19) a) Multiplying by \( A \) once rotates it by 30 degrees, then multiplying by \( A \) again rotates it by another 30 degrees. Hence, the net effect is a counterclockwise rotation by 60 degrees.

b) This undoes one multiplication by \( A \), so it must be a clockwise rotation by 30 degrees.

c) You first return to the original position when you’ve gone in a complete circle, which is 360 degrees. This takes 12 rotations of 30 degrees, so \( k = 12 \).