6.1.1) a) In order to be a linear transformation, we should have \( L(0) = 0 \). However, this gives us \( L([0 0]) = [1 0 0] \neq 0 \), so it is not a linear transformation.

b) Let \([u_1 u_2]\) and \([v_1 v_2]\) be arbitrary vectors in \( \mathbb{R}_2 \), and let \( c \) be an arbitrary real number. For the first part of the definition, we can compute

\[
L([u_1 u_2] + [v_1 v_2]) = L([u_1 + v_1 u_2 + v_2]) = [u_1 + v_1 u_2 + v_2 u_1 + v_1 - u_2 - v_2] = L([u_1 u_2]) + L([v_1 v_2]).
\]

For the second part of the definition, we can compute

\[
L(c[u_1 u_2]) = L([cu_1 cu_2]) = [cu_1 + cu_2 cu_2 cu_1 - cu_2] = c[u_1 + u_2 u_2 u_1 - u_2] = cL([u_1 u_2]).
\]

Therefore, \( L \) is a linear transformation.

6.1.4) a) Let \( A \) and \( B \) be arbitrary \( n \times n \) matrices and let \( c \) be an arbitrary real number. Then

\[
L(A + B) = (A + B)^T = A^T + B^T = L(A) + L(B),
\]

and

\[
L(cA) = (cA)^T = cA^T = cL(A),
\]

so \( L \) is a linear transformation.

b) \( L \) is not a linear transformation, as it fails the scalar multiplication condition. For example,

\[
L(2I_n) = (2I_n)^{-1} = \frac{1}{2} I_n \neq 2I_n = 2(I_n)^{-1} = 2L(I_n).
\]

6.1.22) Let \( u, v \in V \) and \( c \in \mathbb{R} \). We can compute \( I(u + v) = u + v = I(u) + I(v) \) and \( I(cu) = cu = cI(u) \).

6.2.12) a) By Theorem 6.6, \( \dim \ker L + \dim \text{range } L = \dim V \). Since \( \dim \ker L \geq 0 \), we must have \( \dim \text{range } L \leq \dim V \).

b) If \( L \) is onto, then \( \text{range } L = W \). Hence, \( \dim \text{range } L = \dim W \). Apply part (a) to get \( \dim W = \dim \text{range } L \leq \dim V \).

6.3.8) c) This problem is purely computational, but some students had difficulty with this sort of problems, so I’ll work out an example.
The representation with respect to $S$ and $T$ means we use $S$ as the basis for the inputs into $L$ and $T$ as the basis for the outputs. We can compute

\[
L \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}
\]

\[
L \left( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}
\]

\[
L \left( \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}
\]

\[
L \left( \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix}
\]

We must write the matrices on the right side as linear combinations of the matrices of $T$. We can apply the isomorphism between $M_{22}$ and $\mathbb{R}^4$ that sends the standard basis for one to the other to get that this is equivalent to writing the vectors

\[
\begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 0 \\ 0 \\ 4 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

as linear combinations of

\[
\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

which are the vectors in the basis $T$.

If we wish to write one vector as a linear combination of others, we make the one vector $\mathbf{b}$, the other vectors the columns of $A$, and solve $A\mathbf{x} = \mathbf{b}$. The coordinates of $\mathbf{x}$ are what we are looking for. We can do this using the augmented matrix $[A|\mathbf{b}]$. In this case, we have the same matrix $A$ for four different augmented vectors $\mathbf{b}$, we can save time by doing them all at once. We make a matrix with the relevant vectors as columns, and then put it in reduced row echelon form.

\[
\begin{bmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 2 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 2 \\
0 & 0 & 1 & 0 & 3 & 0 & 4 & 0 \\
1 & 0 & 0 & 0 & 0 & 3 & 0 & 4 \\
\end{bmatrix}
\]

swap

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 3 & 0 & 4 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 2 \\
0 & 0 & 1 & 0 & 3 & 0 & 4 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 2 & 0 \\
\end{bmatrix}
\]

$-R1$

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 3 & 0 & 4 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 2 \\
0 & 0 & 1 & 0 & 3 & 0 & 4 & 0 \\
0 & 1 & 1 & 0 & -3 & 2 & -4 & 0 \\
\end{bmatrix}
\]

$-R2$
Thus, the coordinates of $L \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right)$ with respect to $T$ are $\begin{bmatrix} 0 \\ -2 \\ 3 \\ 2 \end{bmatrix}$, so this is the first column of the matrix for $L$ with respect to $S$ and $T$. This works similarly for the columns, and the matrix we seek is

$$\begin{bmatrix}
0 & 3 & 0 & 4 \\
-2 & -3 & -2 & -4 \\
3 & 0 & 4 & 0 \\
2 & 4 & 2 & 6
\end{bmatrix}$$

You weren’t expected to write the exposition as part of your homework, of course, but just the computations.

6.5.1) a) If we take $P = I$, then $A = I^{-1}AI$.

b) If $B$ is similar to $A$, then there is an invertible matrix $P$ such that $B = P^{-1}AP$. This gives us $PBP^{-1} = PP^{-1}APP^{-1} = A$. Let $Q = P^{-1}$ and we have $A = Q^{-1}BQ$, which says that $A$ is similar to $B$.

c) If $C$ is similar to $B$, then there is an invertible matrix $P$ such that $C = P^{-1}BP$. If $B$ is similar to $A$, then there is an invertible matrix $Q$ such that $B = Q^{-1}AQ$. Let $R = QP$. Substitute to get $C = P^{-1}BP = P^{-1}Q^{-1}AP = (QP)^{-1}A(QP) = R^{-1}AR$, which says that $C$ is similar to $A$.

6.5.6) If $B$ is similar to $A$, then there is an invertible matrix $P$ such that $B = P^{-1}AP$. We can compute

$$B^k = BB \ldots B = (P^{-1}AP)(P^{-1}AP) \ldots (P^{-1}AP) = P^{-1}A(PP^{-1})A(PP^{-1}) \ldots (PP^{-1})AP = P^{-1}AA \ldots AP = P^{-1}A^kP,$$

from which $B^k$ is similar to $A^k$. 

3
6.5.11) If $B$ is similar to $A$, then there is an invertible matrix $P$ such that $B = P^{-1}AP$. If $A$ is invertible, we can compute

$$B(P^{-1}A^{-1}P) = (P^{-1}AP)(P^{-1}A^{-1}P) = P^{-1}A(PP^{-1})A^{-1}P = P^{-1}AA^{-1}P = P^{-1}P = I.$$ 

Hence, $B$ is invertible, which settles part (a), and $B^{-1} = P^{-1}A^{-1}P$. This immediately gives us that $B^{-1}$ and $A^{-1}$ are similar, which settles part (b).