1. Find bases for the row space and column space of $A$, as well as the rank of $A$, where

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ -2 & 1 & 7 & 2 \\ -1 & -4 & -1 & 3 \\ 3 & 2 & -7 & -1 \end{bmatrix}$$

We start by putting the matrix in row echelon form.

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ -2 & 1 & 7 & 2 \\ -1 & -4 & -1 & 3 \\ 3 & 2 & -7 & -1 \end{bmatrix} + 2R_1 \quad +1R_1 \quad -3R_1$$

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 5 & 5 & 10 \\ 0 & -2 & -2 & 7 \\ 0 & -4 & -4 & -13 \end{bmatrix} + 1R_2$$

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 5 & 5 & 10 \\ 0 & -2 & -2 & 7 \\ 0 & 1 & 1 & -3 \end{bmatrix} \text{ swap}$$

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 1 & -3 \\ 0 & -2 & -2 & 7 \\ 0 & 5 & 5 & 10 \end{bmatrix} + 2R_2 \quad -5R_2$$

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 25 \end{bmatrix} -25R_3$$

Row operations do not change the row space, so the rows of the matrix at the end have the same span as those of $A$. Furthermore, the nonzero rows of a matrix in row echelon form are linearly independent. Therefore, the row space has a basis $\{[1 \ 2 \ -1 \ 4], \ [0 \ 1 \ 1 \ -3], \ [0 \ 0 \ 0 \ 1]\}$.

From the final matrix, it is clear that the first, second, and fourth columns of the matrix are the pivot columns. Thus, the first, second, and fourth columns of the original matrix form a basis for the column space.
Finally, the row space and column space each have bases with three vectors, so they have dimension three. Therefore, the rank of $A$ is 3.

This problem was intended to be easy, and it was.
2. Let $W$ be a subspace of $\mathbb{R}^3$ with basis

$$\left\{ \begin{bmatrix} 1 \\ 8 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix} \right\}.$$ 

Find an orthonormal basis for $W$.

We do the Gram-Schmidt algorithm to find an orthonormal basis.

$$v_1 = u_1 = \begin{bmatrix} 1 \\ 8 \\ 4 \end{bmatrix}$$  

$$||v_1|| = \sqrt{1^2 + 8^2 + 4^2} = 9$$

$$w_1 = \frac{v_1}{||v_1||} = \frac{1}{9} \begin{bmatrix} 1 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{9} \\ \frac{8}{9} \\ \frac{4}{9} \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}$$

$$(u_2, w_1) = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = (2) \frac{1}{9} + (6) \frac{8}{9} + (3) \frac{4}{9} = \frac{2}{9} + \frac{48}{9} + \frac{12}{9} = \frac{62}{9}$$

$$v_2 = u_2 - (u_2, w_1)w_1 = \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix} - \frac{62}{9} \begin{bmatrix} \frac{1}{9} \\ \frac{8}{9} \\ \frac{4}{9} \end{bmatrix} = \begin{bmatrix} \frac{100}{81} \\ \frac{81}{81} \\ \frac{81}{81} \end{bmatrix} = \frac{5}{81} \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$$||v_2|| = \frac{5}{81} \sqrt{(20)^2 + (-2)^2 + (-1)^2} = \frac{5}{81} \sqrt{400 + 4 + 1} = \frac{5}{81} \sqrt{405} = \frac{5\sqrt{5}}{9}$$

$$w_2 = \frac{v_1}{||v_1||} = \frac{9}{5\sqrt{5}} \begin{bmatrix} 100 \\ 81 \\ 81 \end{bmatrix} = \begin{bmatrix} \frac{20}{9\sqrt{5}} \\ \frac{9\sqrt{5}}{9\sqrt{5}} \\ \frac{9\sqrt{5}}{9\sqrt{5}} \end{bmatrix}$$

Thus, our answer is

$$\left\{ \begin{bmatrix} \frac{1}{9} \\ \frac{8}{9} \\ \frac{4}{9} \end{bmatrix}, \begin{bmatrix} \frac{20}{9\sqrt{5}} \\ \frac{9\sqrt{5}}{9\sqrt{5}} \\ \frac{9\sqrt{5}}{9\sqrt{5}} \end{bmatrix} \right\}$$
I realize that the arithmetic was messy here. I tried to simplify the arithmetic for you by ensuring that \[
\begin{pmatrix}
  100 \\
  -10 \\
  -5
\end{pmatrix}
\] had an obvious factor of 5, but most of the class didn’t catch the factor of 5. Still, most of the class managed to get most of the points, which is pretty good, and better than I expected for this problem.
3. Find a basis for the null space of the matrix

\[ A = \begin{bmatrix} 3 & 8 & 4 & -1 & -3 \\ 1 & 2 & -2 & -1 & 3 \end{bmatrix} \]

The null space of \( A \) is the set of solutions to \( Ax = 0 \). We solve this by putting \( A \) into reduced row echelon form.

\[
A = \begin{bmatrix} 3 & 8 & 4 & -1 & -3 \\ 1 & 2 & -2 & -1 & 3 \end{bmatrix} \text{ swap}
\]

\[
A = \begin{bmatrix} 1 & 2 & -2 & -1 & 3 \\ 3 & 8 & 4 & -1 & -3 \end{bmatrix} \text{ swap}
\]

\[
A = \begin{bmatrix} 1 & 2 & -2 & -1 & 3 \\ 3 & 8 & 4 & -1 & -3 \end{bmatrix} -3 \text{ R1}
\]

\[
A = \begin{bmatrix} 1 & 2 & -2 & -1 & 3 \\ 0 & 2 & 10 & 2 & -12 \end{bmatrix} \times \frac{1}{2}
\]

\[
A = \begin{bmatrix} 1 & 2 & -2 & -1 & 3 \\ 0 & 1 & 5 & 1 & -6 \end{bmatrix} -2 \text{ R2}
\]

The first two columns are the pivot columns, and the rest of the columns can be any real number. Thus, we define \( x_3 = r \), \( x_4 = s \), and \( x_5 = t \). The two equations give us

\[
x_1 = 12r + 3s - 15t \quad \text{and} \quad x_2 = -5r - s + 6t.
\]

Thus, we have

\[
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 12r + 3s - 15t \\ -5r - s + 6t \\ r \\ s \\ t \end{bmatrix} = r \begin{bmatrix} 12 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -15 \\ 6 \\ 0 \\ 0 \\ 1 \end{bmatrix}.
\]

Thus, our basis is

\[
\begin{bmatrix} 12 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -15 \\ 6 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

This problem was intended to be easy, and most of the class could get most of the way to putting the matrix in reduced row echelon form. About \( \frac{1}{3} \) of the class got full credit, and some others came close. Unfortunately, a lot of students had no idea what to do after putting the matrix in reduced row echelon form, and quite a few students gave answers consisting of vectors in \( \mathbb{R}^2 \), not \( \mathbb{R}^5 \).

The null space of \( A \) is the set of solutions to \( Ax = 0 \). As \( A \) is a \( 2 \times 5 \) matrix, in order for \( Ax \) to be defined, we must have \( x \in \mathbb{R}^5 \).
4. Let $V$ be a vector space and let $w_1, w_2, w_3 \in V$. Suppose that \( \{w_1, w_2, w_3\} \) is a linearly independent set. Must \( \{w_1 + w_2, w_2 + w_3, w_3 + w_1\} \) also be a linearly independent set? Either prove that the latter set is always linearly independent, or else give a counterexample.

Suppose that the latter set is linearly dependent. Then there are constants $c_1, c_2,$ and $c_3$, not all zero, such that
\[
c_1(w_1 + w_2) + c_2(w_2 + w_3) + c_3(w_3 + w_1) = 0.
\]
We can rearrange this as
\[
(c_1 + c_3)w_1 + (c_1 + c_2)w_2 + (c_2 + c_3)w_3 = 0.
\]
Because \( \{w_1, w_2, w_3\} \) is linearly independent, the only nontrivial linear combination of its vectors that gives 0 is for all of the coefficients to be zero. Thus, we must have $c_1 + c_3 = 0$, $c_1 + c_2 = 0$, and $c_2 + c_3 = 0$. Solving this system of linear equations gives us $c_1 = -c_3 = c_2 = -c_1$, from which $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$, so the set \( \{w_1 + w_2, w_2 + w_3, w_3 + w_1\} \) is linearly independent.

While the above solution was the intended one, and all but one of the students that got the problem mostly correct roughly followed it, there were other solutions. One student found roughly the following idea.

Let $W$ be a subspace of $V$ consisting of the span of \( \{w_1, w_2, w_3\} \). Let $X$ be a subspace of $V$ consisting of the span of \( \{w_1 + w_2, w_2 + w_3, w_3 + w_1\} \). We can compute
\[
w_1 = \frac{1}{2}(w_1 + w_2) - \frac{1}{2}(w_2 + w_3) + \frac{1}{2}(w_3 + w_1)
w_2 = \frac{1}{2}(w_1 + w_2) + \frac{1}{2}(w_2 + w_3) - \frac{1}{2}(w_3 + w_1)
w_3 = -\frac{1}{2}(w_1 + w_2) + \frac{1}{2}(w_2 + w_3) + \frac{1}{2}(w_3 + w_1)
\]
Because $X$ is a subspace of $V$, any linear combination of vectors in $X$ is also in $X$, so $w_1, w_2, w_3 \in X$. This is a set of three linearly independent vectors in $X$, so by Theorem 4.11, it can be extended to a basis for $X$. Therefore, $X$ has a basis consisting of at least three vectors, so the dimension of $X$ is at least three.

By Corollary 4.5, a set of three vectors cannot span a vector space of dimension greater than three. By definition, \( \{w_1 + w_2, w_2 + w_3, w_3 + w_1\} \) spans $X$, so the dimension of $X$ is three. Theorem 4.12 tells us that because \( \{w_1 + w_2, w_2 + w_3, w_3 + w_1\} \) is a set of three vectors that spans a vector space $X$ of dimension three, the set is a basis for $X$, and therefore is linearly independent.
In the above proof, once you show that $w_1, w_2, w_3 \in X$, there are quite a few ways to finish the proof.

A number of students set $V = \mathbb{R}^3$ and picked three vectors in $\mathbb{R}^3$ and showed that the specified linear combinations of them are also linearly independent. It is possible to make something along these lines work, but even ignoring the computational part, it’s still a much harder proof than either of the above. Students who tried this offered no attempt at showing that it worked for all choices of $V$ and $w_1, w_2, w_3 \in V$, and thus got very little credit.

Let $S = \{w_1, w_2, w_3\}$ and $W$ be a subspace of $V$ consisting of the span of $S$. Since this is a linearly independent set that spans $W$ by definition, $S$ is a basis for $W$. Hence, for any $u \in W$, we can write $[u]_S$, the coordinates for $u$ with respect to the basis $S$. The proof of Theorem 4.14 shows that the function $L : W \to \mathbb{R}^3$ defined by $L(u) = [u]_S$ is an isomorphism.

We can compute $L(w_1 + w_2) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $L(w_2 + w_3) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $L(w_3 + w_1) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. We can also compute

$$\det \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = 2 \neq 0,$$

so by the theorem at the bottom of page 281, the columns of the matrix are linearly independent. This means that $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a linearly independent set, so if we have constants $c_1, c_2, \text{ and } c_3$ such that

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0,$$

then $c_1 = c_2 = c_3 = 0$.

Suppose we have constants $c_1, c_2, \text{ and } c_3$ such that

$$c_1(w_1 + w_2) + c_2(w_2 + w_3) + c_3(w_3 + w_1) = 0.$$

Then we have

$$0 = L(0) = L(c_1(w_1 + w_2) + c_2(w_2 + w_3) + c_3(w_3 + w_1)) = c_1 L(w_1 + w_2) + c_2 L(w_2 + w_3) + c_3 L(w_3 + w_1) = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$
As we showed above, \[
\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}
\] is linearly independent, so \(c_1 = c_2 = c_3 = 0\). Therefore, \(\{w_1+w_2, w_2+w_3, w_3+w_1\}\) is also linearly independent.

This third proof shows the utility of the concept of isomorphisms of vector spaces. Rather than having three vectors in a vector space in the abstract, we can show that it is equivalent to three particular vectors in a vector space of our choice. However, just having the matrix
\[
\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}
\]
floating around at some point wasn’t useful in itself, as you had to also say quite a bit about isomorphisms if you wanted to go that route. No one mentioned the concept of an isomorphism, and I don’t blame you for that, as this third proof is much harder than the first two.

While I expected this problem to be hard, I was disappointed with how students did. Several students had a mostly correct solution, which is roughly what I expected. The problem is that most of the rest of the class could not get so far as to write down what it means for \(\{w_1, w_2, w_3\}\) to be a linearly independent set.
5. Let $V$ be a vector space and let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in V$. Let $T_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$, $T_3 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and $T_4 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Let $W_2$ be the span of $T_2$, $W_3$ be the span of $T_3$, and $W_4$ be the span of $T_4$. Suppose that $T_2$ is a linearly independent set, but $T_3$ and $T_4$ are not, and that $T_4$ spans $V$, but $T_2$ and $T_3$ do not. Find the dimensions of $W_2$, $W_3$, $W_4$, and $V$.

$T_2$ spans $W_2$ and is linearly independent, so $T_2$ is a basis for $W_2$. There are two vectors in $T_2$, so the dimension of $W_2$ is 2.

$W_3$ contains $W_2$, so the dimension of $W_3$ is at least two. Since $W_3$ is spanned by $T_3$, a set of three vectors, it has dimension at most three. If the dimension of $W_3$ were three, then by Theorem 4.12, a set of three vectors would be linearly independent if and only if it spanned $W_3$. However, $T_3$ spans $W_3$ by definition, but $T_3$ is not linearly independent. Therefore, $W_3$ does not have dimension three, so it must have dimension 2.

By the same argument, $W_4$ cannot have dimension 4, because $T_4$ is a set of four vectors that spans $W_4$ but is not linearly independent. Therefore, the dimension of $W_4$ is at most three. The dimension of $W_4$ is at least two, because it contains $W_3$, which has dimension two. Because $T_4$ spans $V$, $V = W_4$.

If $W_4$ had dimension two, then $V$ would have dimension 2. By Theorem 4.12, a linearly independent set of two vectors in a vector space of dimension two is a basis. However, $T_2$ is a linearly independent set of two vectors that does not span $V$, so it is not a basis for $V$. Therefore, $W_4$ does not have dimension 2, so it must have dimension 3. Hence, the dimension of $V$ is also 3.