Math 231, Preliminary Exam 2

April 13, 2007

Name:

Show your work and explain your solutions.

No calculators, books, or notes.

There are 5 questions and 6 pages in this exam. (If you need more paper, just ask.)

Question	Score
1	
2	
3	
4	
5	
Total (out of 65)	

Problem 1. (15 points)

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -2 \\ -1 & 1 & 3 \end{bmatrix}$.

(a) Find the eigenvalues of A.

(b) Find the eigenspaces of A.

(c) Is A diagonalizable?
If so, explain why you can diagonalize and find a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$.
If not, explain why you cannot diagonalize.

(a) The characteristic polynomial of A is

$$\begin{aligned} det(A - \lambda I) = det \begin{bmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & -2 \\ -1 & 1 & 3-\lambda \end{bmatrix} &= (1-\lambda) det \begin{bmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix} \\ &= (1-\lambda)((-\lambda)(3-\lambda) - (-2)(1)) \\ &= (1-\lambda)(\lambda^2 - 3\lambda + 2) \\ &= (1-\lambda)(\lambda - 1)(\lambda - 2) \end{aligned}$$

So the eigenvalues of A are 1 (with multiplicity 2) and 2 (with multiplicity 1).

(b) The eigenspace for eigenvalue 1:
I will solve the system $A - 1I = O$ by row reducing $A - 1I$.

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & -2 \\ -1 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore this eigenspace is $s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

The eigenspace for eigenvalue 2:
I will solve the system $A - 2I = O$ by row reducing $A - 2I$.

$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & -2 \\ -1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore this eigenspace is $s \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

(c) The work in part (b) gives us three linearly independent eigenvectors and so A is diagonalizable.

In fact, $A = PDP^{-1}$ for $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

Problem 2. (15 points)

Suppose each year the population of the United States of American and Canada migrate as follows: 6% of those in Canada move to the USA and 2% of those in the USA move to Canada. Assume no one dies and no one moves in or out of the USA or Canada from elsewhere. Currently, the population of the USA is approximately 300,000,000 and the population of Canada is approximately 30,000,000.

Let $v_k = \begin{bmatrix} x_k \\ y_k \end{bmatrix}$ where x_k denotes the population of the USA in k years and y_k denotes the population of Canada in k year.

(a) What matrix relates v_k to the current distribution v_0. Give the precise relationship.

(b) Why must 1 be an eigenvalue of this matrix?

(c) Explain how you can use the matrix from part (a) to find a formula for the population of the USA and Canada in k years. (Don't actually do it.)

(d) Find v_∞ the eventual distribution of the population in the USA and Canada.

(e) Does this distribution depend on the current distribution of the population?

(a) If $A = \begin{bmatrix} .98 & .06 \\ .02 & .94 \end{bmatrix}$, then $v_k = A^k v_0$.

(b) A is a Markov matrix and therefore have an eigenvalue equal to 1.

(c) Diagonalize A so that you have $A = PDP^{-1}$. Then $A^k = PD^kP^{-1}$ and D^k is simple to calculate. We simply have to take the power of the entries along the diagonal. This gives $v_k = PD^kP^{-1}v_0$.

(d) The eventual distribution, v_∞, is an eigenvector of A with eigenvalue 1. Moreover the entries of v_∞ add up to 330,000,000.
So we solve $A - 1I$ to find the eigenvectors of A with eigenvalue 1.

$$A - I = \begin{bmatrix} -.02 & .06 \\ .02 & -.06 \end{bmatrix} \longrightarrow \begin{bmatrix} -.02 & .06 \\ 0 & 0 \end{bmatrix}$$

So the eigenspace is $s\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Therefore the eventual distribution is $\begin{bmatrix} \frac{3}{4}330,000,000 \\ \frac{1}{4}330,000,000 \end{bmatrix}$.

(e) No. It only depends on the matrix A and the total population 330,000,000.

Problem 3. (15 points)

Let $W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \text{ such that } 2a + 4c - d = 0 \text{ and } b - 2d = 0 \right\}$.

(a) Show that W is a subspace of $\mathbb{R}^4$.
Make sure to state the result you are using the show this.
Hint: There is a matrix A such that W is one of the subspaces associated with A.

(b) Find a basis for W.

(c) What is the dimension of W?

(d) Why is $\left\{ \begin{bmatrix} -5 \\ 4 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 8 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \\ 2 \end{bmatrix} \right\}$ not a basis for W?
The definition of basis consists of two conditions; which of these conditions is not satisfied?

(a) $W = N(A)$ where $A = \begin{bmatrix} 2 & 0 & 4 & -1 \\ 0 & 1 & 0 & -2 \end{bmatrix}$. But the null space of a matrix is always a subspace, therefore W is a subspace.

(b) We row reduce A.

$$A = \begin{bmatrix} 2 & 0 & 4 & -1 \\ 0 & 1 & 0 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & -\frac{1}{2} \\ 0 & 1 & 0 & -2 \end{bmatrix}$$

Therefore $N(A) = s \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{2} \\ 2 \\ 0 \\ 1 \end{bmatrix}$ and a basis for $W = N(A)$ is $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(c) The dimension of W is 2.

(d) This set has 3 vectors whereas a basis must have 2 since the dimension is 2.
Since there are too many vectors, they must not be linearly independent.

Problem 4. (10 points)

Let $A \in M_{m\times n}(\mathbb{R})$ and suppose $A^2 = O$.

(a) Show that the column space of A is contained in the null space of A.

(b) Show that $\text{rank}(A) \leq \frac{n}{2}$.

(a) The column space of A can be characterized as $C(A) = \{Ax \text{ such that } x \in \mathbb{R}^n\}$.
If $y \in C(A)$, then $y = Ax$ for some x.
Therefore, $Ay = AAx = A^2x = Ox = 0$
and by definition of the null space of A, $y \in N(A)$.
This implies that the column space of A is containted in the null space of A.

(b) By part (a), since the column space of A is contained in the null space of A,
we have $\dim(C(A)) \leq \dim(N(A))$.
In other words, $\text{rank}(A) \leq \text{nullity}(A)$.
By the rank-nullity theorem, $\text{rank}(A) + \text{nullity}(A) = n$, the number of columns of A.
So $\text{rank}(A) \leq \frac{n}{2}$.

Problem 5. (10 points)

(a) Define linear transformation.

(b) Which of the following transformations are linear? If a transformation is not linear, say why.

$$Q : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \qquad Q \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x+z \\ xy+z \\ y-z \end{bmatrix}$$

$$R : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \qquad R \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 1 \\ 0 & 2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$S : P_4 \longrightarrow P_4 \qquad S(f(x)) = f'(x) + 1$$

(a) A linear transformation is a function $T : V \to W$ from a vector space V to a vector space W that satisfies

i. $T(u+v) = T(u) + T(v)$, and

ii. $T(cu) = cT(u)$

for all vectors $u, v \in V$ and all scalars c.

(b) Q is not linear because of the xy term in the second row.

Note that $Q \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ and $Q \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ but $Q \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

R is multiplication of the left by a matrix and hence is linear.

S is not linear because of the 1. In particular, S of the zero function is 1 not 0.