

22. The Gram-Schmidt algorithm begins

$$f_1 = u_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix},$$

$$f_2 = u_2 - \frac{u_2 \cdot f_1}{f_1 \cdot f_1} f_1 = u_2 - \frac{1}{2} f_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \end{bmatrix},$$

which, for convenience, we replace with $f_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}$. Continuing,

$$f_3 = u_3 - \frac{u_3 \cdot f_1}{f_1 \cdot f_1} f_1 - \frac{u_3 \cdot f_2}{f_2 \cdot f_2} f_2 = u_3 - \frac{1}{2} f_1 - \frac{1}{6} f_2 = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ -1 \end{bmatrix},$$

which we replace with $f_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}$. The vectors f_1, f_2, f_3 provide an orthogonal basis for V .

We obtain an orthonormal basis by dividing each of these vectors by its length, obtaining

$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \frac{1}{12} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}$. This basis is far from unique. Applying Gram-Schmidt to

the given vectors in a different order, or to other spanning sets for V , leads to infinitely many other orthonormal bases.