HW9

3.3

2. (b) v_1 is not an eigenvector: an eigenvector is, by definition, **not zero**.

$Av_2 = \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix}$ is not λv_2 for any scalar λ, so v_2 is not an eigenvector.

$Av_3 = v_3$, so v_3 is an eigenvector of A corresponding to $\lambda = 1$.

$Av_4 = -2v_4$, so v_4 is an eigenvector of A corresponding to $\lambda = -2$.

$Av_5 = 5v_5$, so v_5 is an eigenvector of A corresponding to $\lambda = 5$.

$Av_6 = \begin{bmatrix} 12 \\ 8 \\ 6 \end{bmatrix}$ is not λv_6 for any scalar λ, so v_6 is not an eigenvector.

6 (b) Call the given matrix A. The characteristic polynomial of A is $\begin{vmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{vmatrix} = -\lambda(-1-\lambda) - 6 = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$, so $\lambda = -3$ and $\lambda = 2$ are the eigenvalues of A. To find the eigenspace corresponding to $\lambda = -3$, we must solve the homogeneous system $(A - \lambda I)x = 0$ with $\lambda = -3$. We have

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \to \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}.$$

The variable $x_2 = t$ is free and $2x_1 = -3x_2 = -3t$, so $x_1 = -\frac{3}{2}t$. The eigenspace corresponding to $\lambda = -3$ is the set of vectors of the form of $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}$.

To find the eigenspace corresponding to $\lambda = 2$, we must solve the homogeneous system $(A - \lambda I)x = 0$ with $\lambda = 2$. We have

$$A - \lambda I = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

Again, $x_2 = t$ is free, but $x_1 = x_2 = t$. The eigenspace corresponding to $\lambda = 2$ is the set of vectors of the form of $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(h) Call the matrix A. Then $A - \lambda I = \begin{bmatrix} 1-\lambda & 6 & 0 & 0 \\ 5 & 2-\lambda & 0 & 0 \\ 0 & 0 & 3-\lambda & -2 \\ 0 & 0 & 1 & -\lambda \end{bmatrix}$ and the characteristic polynomial of A is $\det(A-\lambda I) = \lambda^4 - 6\lambda^3 - 17\lambda^2 + 78\lambda - 56 = (\lambda-1)(\lambda-2)(\lambda-7)(\lambda+4)$. There are four eigenvalues, $\lambda = 1$, $\lambda = 2$, $\lambda = 7$, and $\lambda = -4$. To find the eigenspace for $\lambda = 1$, we must solve the homogeneous system $(A - \lambda I)x = 0$ with $\lambda = 1$. We have

$$A - \lambda I = \begin{bmatrix} 0 & 6 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so $x_4 = t$ is free, $x_3 = x_4 = t$, $x_2 = 0$, and $x_1 = -\frac{1}{5}x_2 = 0$. The eigenspace consists of vectors $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. To find the eigenspace for $\lambda = 2$, we must solve the homogeneous system $(A - \lambda I)\mathsf{x} = 0$ with $\lambda = 2$. We have

$$A - \lambda I = \begin{bmatrix} -1 & 6 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -6 & 0 & 0 \\ 0 & 30 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so $x_4 = t$ is free, $x_3 = 2x_4 = 2t$, $x_2 = 0$, and $x_1 = 6x_2 = 0$. The eigenspace consists of vectors $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2t \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$. To find the eigenspace for $\lambda = 7$, we must solve the homogeneous system $(A - \lambda I)\mathsf{x} = 0$ with $\lambda = 7$. We have

$$A - \lambda I = \begin{bmatrix} -6 & 6 & 0 & 0 \\ 5 & -5 & 0 & 0 \\ 0 & 0 & -4 & -2 \\ 0 & 0 & 1 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so $x_2 = t$ is free, $x_4 = 0$, $x_3 = 7x_4 = 0$, and $x_1 = x_2 = t$. The eigenspace consists of vectors $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. To find the eigenspace for $\lambda = -4$, we must solve the homogeneous system $(A - \lambda I)\mathsf{x} = 0$ with $\lambda = -4$. We have

$$A - \lambda I = \begin{bmatrix} 5 & 6 & 0 & 0 \\ 5 & 6 & 0 & 0 \\ 0 & 0 & 7 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 5 & 6 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -30 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so $x_2 = t$ is free, $x_4 = 0$, $x_3 = -4x_4 = 0$, and $5x_1 = -6x_2 = -6t$, so $x_1 = -\frac{6}{5}$. The eigenspace consists of vectors $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{6}{5}t \\ t \\ 0 \\ 0 \end{bmatrix} = t\begin{bmatrix} -\frac{6}{5} \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

15. (a) We are given $A\mathsf{x} = \lambda\mathsf{x}$ with $\mathsf{x} \neq 0$. Since A is invertible, $A\mathsf{x} \neq 0$, so $\lambda \neq 0$.
    (b) Multiplying $A\mathsf{x} = \lambda\mathsf{x}$ by A^{-1} gives $\mathsf{x} = \lambda A^{-1}\mathsf{x}$, so $A^{-1}\mathsf{x} = \frac{1}{\lambda}\mathsf{x}$. Thus x is also an eigenvector of A^{-1}, with eigenvalue $\frac{1}{\lambda}$.

18. (a) [BB] The answer is yes. Since $A\mathsf{v} = \lambda\mathsf{v}$, $(5A)\mathsf{v} = 5\lambda\mathsf{v} = (5\lambda)\mathsf{v}$, so v is an eigenvector of $5A$ with eigenvalue 5λ.
    (b) The answer is yes. Since $A\mathsf{v} = \lambda\mathsf{v}$, $(5A)\mathsf{v} = 5\lambda\mathsf{v} = \lambda(5\mathsf{v})$, so $5\mathsf{v}$ is an eigenvector of $5A$ with eigenvalue λ.
    (c) The answer is yes, and this is a consequence of the fact that eigenvectors correspond to fixed lines—see 3.3.4 and Exercise 9. Since $A\mathsf{v} = \lambda\mathsf{v}$, $A(5\mathsf{v}) = 5A\mathsf{v} = 5(\lambda\mathsf{v}) = \lambda(5\mathsf{v})$, so $5\mathsf{v}$ is an eigenvector of A with eigenvalue λ.