

- 1 (b) This system is homogeneous. The elementary row operations will not change the right hand column of 0s, so we omit these in performing Gaussian elimination (but remember they are there)!

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -7 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The variable $z = t$ is free. Back substitution gives $y + z = 0$, so $y = -z = -t$ and $x - 2y + z = 0$, so $x = 2y - z = -3t$. The solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$.

- 3 (d) $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$ is $Ac = 0$ with $A = \begin{bmatrix} 1 & -3 & 4 & 1 \\ 2 & 0 & 2 & 1 \\ 3 & 1 & 2 & 13 \end{bmatrix}$ and $c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$.

Gaussian elimination proceeds

$$A \rightarrow \begin{bmatrix} 1 & -3 & 4 & 1 \\ 0 & 6 & -6 & -1 \\ 0 & 10 & -10 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 4 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 4 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since c_3 is a free variable, this homogeneous system has nontrivial solutions. For example, $-v_1 + v_2 + v_3 + 0v_4 = 0$. The vectors are linearly dependent.

- (e) $c_1v_1 + c_2v_2 + c_3v_3 = 0$ is $Ac = 0$ with $A = \begin{bmatrix} 4 & 8 & 4 \\ 5 & 6 & 2 \\ 1 & 10 & -3 \\ 7 & 8 & 4 \end{bmatrix}$ and $c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$. Gaussian

elimination proceeds

$$A \rightarrow \begin{bmatrix} 1 & 10 & -3 \\ 0 & -32 & 16 \\ 0 & -44 & 17 \\ 0 & -62 & 25 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 10 & -3 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & -5 \\ 0 & 0 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 10 & -3 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The solution is $c_3 = c_2 = c_1 = 0$. The vectors are linearly independent.

2.4

(c) The given system is $Ax = b$ where

$$A = \begin{bmatrix} -1 & 0 & 0 & -1 & 2 \\ 3 & 1 & 2 & 4 & -1 \\ 4 & -1 & 1 & 0 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

The augmented matrix is $\left[\begin{array}{ccccc|c} -1 & 0 & 0 & -1 & 2 & -2 \\ 3 & 1 & 2 & 4 & -1 & 1 \\ 4 & -1 & 1 & 0 & 2 & 1 \end{array} \right]$. Gaussian elimination proceeds

$$\begin{aligned} & \left[\begin{array}{ccccc|c} -1 & 0 & 0 & -1 & 2 & -2 \\ 3 & 1 & 2 & 4 & -1 & 1 \\ 4 & -1 & 1 & 0 & 2 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 2 & 1 & 5 & -5 \\ 0 & -1 & 1 & -4 & 10 & -7 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 2 & 1 & 5 & -5 \\ 0 & 0 & 3 & -3 & 15 & -12 \end{array} \right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 2 & 1 & 5 & -5 \\ 0 & 0 & 1 & -1 & 5 & -4 \end{array} \right]. \end{aligned}$$

Variables $x_4 = t$ and $x_5 = s$ are free. Back substitution yields $x_3 - x_4 + 5x_5 = -4$, so $x_3 = -4 + x_4 - 5x_5 = -4 + t - 5s$; $x_2 + 2x_3 + x_4 + 5x_5 = -5$, so $x_2 = -5 - 2x_3 - x_4 - 5x_5 = -5 - 2(-4 + t - 5s) - t - 5s = 3 - 3t + 5s$, $x_1 + x_4 - 2x_5 = 2$, so $x_1 = 2 - t + 2s$. The solution is

$$x = \begin{bmatrix} 2 - t + 2s \\ 3 - 3t + 5s \\ -4 + t - 5s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -4 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 5 \\ -5 \\ 0 \\ 1 \end{bmatrix}.$$

which is of the form $x_p + x_h$ with $x_p = \begin{bmatrix} 2 \\ 3 \\ -4 \\ 0 \\ 0 \end{bmatrix}$ a particular solution and $x_h = t \begin{bmatrix} -1 \\ -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} +$

$s \begin{bmatrix} 2 \\ 5 \\ -5 \\ 0 \\ 1 \end{bmatrix}$ a solution to the corresponding homogeneous system.

2.4

7. (a) This is $Ax = b$ with

$$A = \begin{bmatrix} 1 & -2 & 3 & 1 & 3 & 4 \\ -3 & 6 & -8 & 2 & -11 & -15 \\ 1 & -2 & 2 & -4 & 6 & 9 \\ -2 & 4 & -6 & -2 & -6 & -7 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix}.$$

$$(b) [A|b] = \left[\begin{array}{cccccc|c} 1 & -2 & 3 & 1 & 3 & 4 & -1 \\ -3 & 6 & -8 & 2 & -11 & -15 & 2 \\ 1 & -2 & 2 & -4 & 6 & 9 & 3 \\ -2 & 4 & -6 & -2 & -6 & -7 & 1 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{cccccc|c} 1 & -2 & 3 & 1 & 3 & 4 & -1 \\ 0 & 0 & 1 & 5 & -2 & -3 & -1 \\ 0 & 0 & -1 & -5 & 3 & 5 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{array} \right] \rightarrow \left[\begin{array}{cccccc|c} 1 & -2 & 3 & 1 & 3 & 4 & -1 \\ 0 & 0 & 1 & 5 & -2 & -3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{array} \right].$$

This is row echelon form. The variables $x_2 = t$ and $x_4 = s$ are free. By back substitution, we get $x_6 = -1$, $x_5 = 3 - 2x_6 = 5$, $x_3 = -1 - 5x_4 + 2x_5 + 3x_6 = -1 - 5s + 10 - 3 = 6 - 5s$, and $x_1 = -1 + 2x_2 - 3x_3 - x_4 - 3x_5 - 4x_6 = -1 + 2t - 18 + 15s - s - 15 + 4 =$

$$-30 + 2t + 14s. \text{ In vector form, the solution is } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -30 + 2t + 14s \\ t \\ 6 - 5s \\ s \\ 5 \\ -1 \end{bmatrix} =$$

$$\begin{bmatrix} -30 \\ 0 \\ 6 \\ 0 \\ 5 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 14 \\ 0 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\text{This is } x_p + x_h \text{ with } x_p = \begin{bmatrix} -30 \\ 0 \\ 6 \\ 0 \\ 5 \\ -1 \end{bmatrix} \text{ and } x_h = t \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 14 \\ 0 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

(c) In part (b), we saw that $A \begin{bmatrix} -30 \\ 0 \\ 6 \\ 0 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$ so, in view of 2.1.33, we have

$$\begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = -30 \begin{bmatrix} 1 \\ -3 \\ 1 \\ -2 \end{bmatrix} + 0 \begin{bmatrix} -2 \\ 6 \\ -2 \\ 4 \end{bmatrix} + 6 \begin{bmatrix} 3 \\ -8 \\ 2 \\ -6 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 2 \\ -4 \\ -2 \end{bmatrix} + 5 \begin{bmatrix} 3 \\ -11 \\ 6 \\ -6 \end{bmatrix} - \begin{bmatrix} 4 \\ -15 \\ 9 \\ -7 \end{bmatrix}.$$

16. (a) The vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ is a linear combination of the given vectors if and only if there exist scalars c_1, c_2, c_3, c_4 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{b}$. This is $A\mathbf{c} = \mathbf{b}$ with

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}. \text{ Gaussian elimination proceeds}$$

$$A \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & b_1 \\ 0 & 0 & 0 & -1 & b_2 - b_1 \\ 0 & 0 & -1 & -1 & b_3 - b_1 \\ 0 & -1 & -1 & -1 & b_4 - b_1 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & b_1 \\ 0 & 1 & 1 & 1 & b_1 - b_4 \\ 0 & 0 & -1 & -1 & b_3 - b_1 \\ 0 & 0 & 0 & -1 & b_2 - b_1 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & b_1 \\ 0 & 1 & 1 & 1 & b_1 - b_4 \\ 0 & 0 & 1 & 1 & b_1 - b_3 \\ 0 & 0 & 0 & 1 & b_1 - b_2 \end{array} \right]. \text{ This system has a solution for any } \mathbf{b}, \text{ so every } \mathbf{b} \text{ is a}$$

linear combination of the given vectors.

26. We apply Gaussian elimination to the augmented matrix of coefficients:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 2 & -1 & 0 & 2 \\ 0 & 1 & 2 & -4 \\ a & b & c & 3 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & -1 & -2 & 4 \\ 0 & 1 & 2 & -4 \\ 0 & b & c-a & 3+a \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (c-a) - 2b & (3+a) + 4b \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & (c-a) - 2b & (3+a) + 4b \\ 0 & 0 & 0 & 0 \end{array} \right].$$

- (a) If $c - a - 2b \neq 0$, row echelon form is $\left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 1 & \frac{3+a+4b}{c-a-2b} \\ 0 & 0 & 0 & 0 \end{array} \right]$ and there is a unique

$$\text{solution: } z = \frac{3+a+4b}{c-a-2b}, x = -4 - 2z, y = -1 - z.$$

- (b) If $c - a - 2b = 0$ and $3 + a + 4b \neq 0$, the third equation says $0 = 3 + a + 4b \neq 0$, a contradiction. There is no solution in this case.

- (c) If $c - a - 2b = 0$ and $3 + a + 4b = 0$, then row echelon form is $\left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right],$
- $z = t$ is a free variable, and there are infinitely many solutions.

27. We have $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 3 & -1 & -2 & 3 \\ -1 & 3 & 6 & k \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -4 & -8 & 0 \\ 0 & 4 & 8 & k+1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & k+1 \end{array} \right].$

- (a) There are no solutions if $k + 1 \neq 0$; that is, if $k \neq -1$.
- (b) There are infinitely many solutions if $k + 1 = 0$; that is, if $k = -1$.

