HW13

4.3

5. (b) Since S contains pairs of vectors neither of which is a multiple of the other, $\dim V \geq 2$. The student may check that the first three vectors are linearly dependent, as are vectors one, two, and four. Thus $\dim V = 2$ and any two of the given four vectors is a basis for V.

   (c) The first three vectors are linearly independent and hence form a basis for R^3. Thus $V = R^3$ and the first three vectors in S for a basis for V.

17. (a) Gaussian elimination on A proceeds

$$A = \begin{bmatrix} -1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 2 \\ 1 & 1 & 0 & 1 \\ 2 & -1 & 3 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U.$$

The nonzero rows of U are a basis for the row space:

$$\begin{bmatrix} 1 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -1 & 2 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}.$$

The dimension of the row space is 3.

(b) The pivot columns are columns one, two, and four. So columns one, two, and four of A provide a basis for the column space of A:

$$\begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

The dimension of the column space is 3.

(c) The null space of A is the set of $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ that satisfy $A\mathsf{x} = 0$. Using the result of part (a), we see that $x_3 = t$ is free, $x_4 = 0$, $x_2 = x_3 - 2x_4 = t$, and $x_1 = -x_3 + x_4 = -t$, so $\mathsf{x} = t\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$. A basis is $\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$. The nullity of A, which is the dimension of the null space, is 1.

1 (b) The Gram-Schmidt algorithm begins

$$f_1 = u_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

$$f_2 = u_2 - \frac{u_2 \cdot f_1}{f_1 \cdot f_1} f_1 = u_2 - \tfrac{2}{3} f_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \tfrac{2}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}.$$

(Note: multiplying f2 by 3 also changes the relation between f2 and u1 & u2. Be careful!!!)

The calculation of f_3 will be easier if we multiply f_2 by 3 obtaining a new $f_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, which is orthogonal to f_1, as it should be. Continuing,

$$f_3 = u_3 - \frac{u_3 \cdot f_1}{f_1 \cdot f_1} f_1 - \frac{u_3 \cdot f_2}{f_2 \cdot f_2} f_2 = u_3 - \tfrac{2}{3} f_1 - \tfrac{5}{6} f_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \tfrac{2}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \tfrac{5}{6}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

(Here too!!!)

which we would probably replace by $2f_3$ giving as our final set of orthogonal vectors $f_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $f_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $f_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

2. (a) Let $u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $u_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ be the given basis for U. We apply the Gram-Schmidt algorithm to find an orthogonal basis. Thus

$$f_1 = u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$f_2 = u_2 - \frac{u_2 \cdot f_1}{f_1 \cdot f_1} f_1 = u_2 - \frac{2}{2} f_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$f_3 = u_3 - \frac{u_3 \cdot f_1}{f_1 \cdot f_1} f_1 - \frac{u_3 \cdot f_2}{f_2 \cdot f_2} f_2 = u_3 - \frac{1}{2} f_1 - \frac{1}{1} f_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

(Here too!!!)

and, multiplying by 2, we replace f_3 by a new $f_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$.