HW12

4.1

16. (a) Since $A\mathbf{u} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \neq \mathbf{0}$, u is not in the null space.

(b) Gaussian elimination proceeds $\begin{bmatrix} 6 & -12 & -5 & 16 & -2 & -53 \\ -3 & 6 & 3 & -9 & 1 & 29 \\ -4 & 8 & 3 & -10 & 1 & 33 \end{bmatrix}$

$$\to \begin{bmatrix} 6 & -12 & -5 & 16 & -2 & -53 \\ 0 & 0 & \frac{1}{2} & -1 & 0 & \frac{5}{2} \\ 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{7}{3} \end{bmatrix} \to \begin{bmatrix} 6 & -12 & -5 & 16 & -2 & -53 \\ 0 & 0 & \frac{1}{2} & -1 & 0 & \frac{5}{2} \\ 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

from which we see that $A\mathbf{x} = \mathbf{b}$ has a solution. Thus b is indeed in the column space. Setting the free variables x_2 and x_4 equal to 0, we obtain one solution $x_1 = -4$, $x_2 = 0$, $x_3 = 5$, $x_4 = 0$, $x_5 = 2$. Thus $A\begin{bmatrix} -4 \\ 0 \\ 5 \\ 0 \\ 2 \end{bmatrix} = \mathbf{b}$; so

$$\mathbf{b} = -4(\text{Column } 1) + 0(\text{Column } 2) + 5(\text{Column } 3) + 0(\text{Column } 4) + 2(\text{Column } 5)$$
$$= -4\begin{bmatrix} 6 \\ -3 \\ -4 \end{bmatrix} + 5\begin{bmatrix} -5 \\ 3 \\ 3 \end{bmatrix} + 2\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

(c) To find the null space, we solve $A\mathbf{x} = \mathbf{0}$ for $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$.

As we have seen, A can be reduced to $\begin{bmatrix} 6 & -12 & -5 & 16 & -2 \\ 0 & 0 & \frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{3} \end{bmatrix}$, so $x_2 = t$ and $x_4 = s$ are free variables. Back substitution gives $x_5 = 0$, $x_3 = 2x_4 = 2s$, $x_1 = 2x_2 - x_4 = 2t - s$.

Thus the null space consists of vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2t - s \\ t \\ 2s \\ s \\ 0 \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

(d) Applying the elimination process to solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, we see that the system can be solved for any b. The column space is R^3.

(e) rank $A = 3$.

17. (a) Since $A\mathbf{u} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \neq \mathbf{0}$, $\mathbf{u}$ is not in the null space of A.

(b) The question asks if $A\mathbf{x} = \mathbf{b}$ has a solution. Applying Gaussian elimination, we obtain

$$\left[\begin{array}{rrrrr|r} 2 & -8 & 7 & -4 & 3 & -4 \\ 6 & -24 & 24 & -7 & 10 & 5 \\ 10 & -40 & 41 & -8 & 24 & 7 \end{array}\right]$$

$$\to \left[\begin{array}{rrrrr|r} 2 & -8 & 7 & -4 & 3 & -4 \\ 0 & 0 & 3 & 5 & 1 & 17 \\ 0 & 0 & 6 & 12 & 9 & 27 \end{array}\right] \to \left[\begin{array}{rrrrr|r} 2 & -8 & 7 & -4 & 3 & -4 \\ 0 & 0 & 3 & 5 & 1 & 17 \\ 0 & 0 & 0 & 2 & 7 & -7 \end{array}\right]$$

from which we see that $A\mathbf{x} = \mathbf{b}$ has a solution. (There are free variables.) Thus $\mathbf{b}$ is indeed in the column space. Setting the free variables x_2 and x_5 equal to 0, we obtain one solution $x_4 = -\frac{7}{2}$, $3x_3 = 17 - 5x_4 = \frac{69}{2}$, so $x_3 = \frac{23}{2}$, $2x_1 = -4 - 7x_3 + 4x_4 = -\frac{197}{2}$, $x_1 = -\frac{197}{4}$. Thus $A\begin{bmatrix} -\frac{197}{4} \\ 0 \\ \frac{23}{2} \\ -\frac{7}{2} \\ 0 \end{bmatrix} = \mathbf{b}$; so

$$\begin{aligned} \mathbf{b} &= -\frac{197}{4}(\text{Column } 1) + 0(\text{Column } 2) + \frac{23}{2}(\text{Column } 3) \\ &\quad - \frac{7}{2}(\text{Column } 4) + 0(\text{Column } 5) \\ &= -\frac{197}{4}\begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix} + 0\begin{bmatrix} -8 \\ -24 \\ -40 \end{bmatrix} + \frac{23}{2}\begin{bmatrix} 7 \\ 24 \\ 41 \end{bmatrix} - \frac{7}{2}\begin{bmatrix} -4 \\ -7 \\ -8 \end{bmatrix} + 0\begin{bmatrix} 3 \\ 10 \\ 24 \end{bmatrix}. \end{aligned}$$

(c) To find the null space, we solve $A\mathbf{x} = \mathbf{0}$ for $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$.

As we have seen, a row echelon form of A is $\begin{bmatrix} 2 & -8 & 7 & -4 & 3 \\ 0 & 0 & 3 & 5 & 1 \\ 0 & 0 & 0 & 2 & 7 \end{bmatrix}$, so $x_2 = t$ and $x_5 = s$ are free. Back substitution gives $2x_4 = -7x_5$, so $x_4 = -\frac{7}{2}s$, $3x_3 = -5x_4 - x_5 = \frac{33}{2}s$, so $x_3 = \frac{11}{2}s$, $2x_1 = 8x_2 - 7x_3 + 4x_4 - 3x_5 = 8t - \frac{111}{2}s$ so $x_1 = 4t - \frac{111}{4}s$. Thus the null space consists of vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4t - \frac{111}{4}s \\ t \\ \frac{11}{2}s \\ -\frac{7}{2}s \\ s \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -\frac{111}{4} \\ 0 \\ \frac{11}{2} \\ -\frac{7}{2} \\ 1 \end{bmatrix}.$$

(d) Applying the elimination process to solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, we see that the system can be solved for any $\mathbf{b}$. The column space is R^3.

(e) $\operatorname{rank} A = 3$.

1. (b) $\dim \mathsf{R}^3 = 3$ and we are given three vectors. These form a basis if and only if they are linearly independent, so we investigate the system $A\mathsf{x} = 0$, where the columns of A are the given vectors. Gaussian elimination proceeds

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -8 & -18 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

There are nontrivial solutions; the vectors are linearly dependent; they do not form a basis.

(f) Five vectors in a vector space of dimension four cannot form a basis.

9. [BB] W is the null space of $A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 1 & 0 & 1 & 3 \\ 2 & 3 & 5 & 9 \end{bmatrix}$. Using Gaussian elimination,

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 1 & 0 & 1 & 3 \\ 2 & 3 & 5 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & -2 & -2 & -2 \\ 0 & -1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so $x_3 = t$ and $x_4 = s$ are free, $x_2 = -t - s$, $x_1 = -2(-t-s) - 3t - 5s = -t - 3s$, and $\mathsf{x} = \begin{bmatrix} -t-3s \\ -t-s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. The vectors $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ are a basis for W: $\dim W = 2$.