HW11

4.1

4\. (d) The question asks whether there exist scalars a, b, and c so that $\begin{bmatrix}5\\3\\4\end{bmatrix} = a\begin{bmatrix}2\\0\\1\end{bmatrix} + b\begin{bmatrix}2\\3\\2\end{bmatrix} + c\begin{bmatrix}-4\\5\\-1\end{bmatrix}$. For this, we would need
$$\begin{aligned}2a + 4b - 4c &= 5\\ 3b + 5c &= 3\\ a + 2b - c &= 4\end{aligned}$$
. This system has the solution $a = -\frac{5}{4}$, $b = \frac{9}{4}$, $c = -\frac{3}{4}$, so the answer is "yes". The given vector is in the column space of the matrix.

(e) The question asks whether there exist scalars a and b so that $\begin{bmatrix}-4\\-6\\1\\0\end{bmatrix} = a\begin{bmatrix}-1\\-5\\-4\\2\end{bmatrix} + b\begin{bmatrix}2\\3\\9\\4\end{bmatrix}$.

For this, we would need $-a+2b = -4$, $-5a+3b = -6$, $-4a+9b = 1$, and $2a+4b = 0$. There are no solutions, so the answer is no. The given vector is not in the column space.

5\. (c) In general, with $\mathsf{b} = \begin{bmatrix}b_1\\b_2\\b_3\\b_4\end{bmatrix}$, we again attempt to solve $A\mathsf{x} = \mathsf{b}$ for $\mathsf{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$. Gaussian elimination applied to the augmented matrix proceeds

$$\left[\begin{array}{rrr|c} 2 & -1 & 4 & b_1\\ 0 & 2 & 3 & b_2\\ -1 & -5 & 1 & b_3\\ 1 & -3 & 2 & b_4\end{array}\right] \to \left[\begin{array}{rrr|c} 1 & -3 & 2 & b_4\\ 0 & 2 & 3 & b_2\\ 0 & 5 & 0 & b_1 - 2b_4\\ 0 & -8 & 3 & b_3 + b_4\end{array}\right] \to \left[\begin{array}{rrr|c} 1 & -3 & 2 & b_4\\ 0 & 1 & \frac{3}{2} & \frac{1}{2}b_2\\ 0 & 5 & 0 & b_1 - 2b_4\\ 0 & -8 & 3 & b_3 + b_4\end{array}\right]$$
$$\to \left[\begin{array}{rrr|c} 1 & -3 & 2 & b_4\\ 0 & 1 & \frac{3}{2} & \frac{1}{2}b_2\\ 0 & 0 & -\frac{15}{2} & b_1 - 2b_4 - \frac{5}{2}b_2\\ 0 & 0 & 15 & b_3 + b_4 + 4b_2\end{array}\right] \to \left[\begin{array}{rrr|c} 1 & -3 & 2 & b_4\\ 0 & 1 & \frac{3}{2} & \frac{1}{2}b_2\\ 0 & 0 & 15 & -2b_1 + 4b_4 + 5b_2\\ 0 & 0 & 15 & b_3 + b_4 + 4b_2\end{array}\right].$$

Thus b is in the column space if and only if
$$-2b_1 + 4b_4 + 5b_2 = b_3 + b_4 + 4b_2; \qquad (1)$$
that is, if and only if $2b_1 - b_2 + b_3 - 3b_4 = 0$. Here's why. If (1) is not true, the last two lines in our final matrix contradict each other and there is no solution to $A\mathsf{x} = \mathsf{b}$. On the other hand, if (1) holds, then one more step in the Gaussian elimination makes the final row zero and we can solve for x_3, x_2, and x_1 by back substitution, just as in part (b).

8\. (c) $\begin{bmatrix}1&2&3&5\\1&0&1&3\\2&3&5&9\end{bmatrix} \to \begin{bmatrix}1&2&3&5\\0&-2&-2&-2\\0&-1&-1&-1\end{bmatrix} \to \begin{bmatrix}1&2&3&5\\0&1&1&1\\0&0&0&0\end{bmatrix}$

There are two pivots, so the rank is 2.

10\. (a) The null space is the set of solutions to $A\mathsf{x} = 0$.
$$\begin{bmatrix}1&0&3\\3&1&16\\5&2&29\end{bmatrix} \to \begin{bmatrix}1&0&3\\0&1&7\\0&2&14\end{bmatrix} \to \begin{bmatrix}1&0&3\\0&1&7\\0&0&0\end{bmatrix}.$$
Thus $x_3 = t$ is free, $x_2 + 7x_3 = 0$, so $x_2 = -7x_3 = -7t$ and $x_1 + 32x_3 = 0$, so $x_1 = -3x_3 = -3t$. The null space is all vectors of the form $\begin{bmatrix}-3t\\-7t\\t\end{bmatrix} = t\begin{bmatrix}-1\\-7\\1\end{bmatrix}$.

(b) The rank of A is the number of pivot columns. From part (a), we see rank $A = 2$.

(c) $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 3 & 1 & 16 & 3 \\ 5 & 2 & 29 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 7 & -3 \\ 0 & 2 & 14 & -6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 7 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Thus $x_3 = t$ is free, $x_2 + 7x_3 = -3$, so $x_2 = -3 - 7t$ and $x_1 + 3x_3 = 2$, so $x_1 = 2 - 3t$. The solution is $\mathsf{x} = \begin{bmatrix} 2-3t \\ -3-7t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ -7 \\ 1 \end{bmatrix}$.

(d) From part (c), we see that one particular solution to $A\mathsf{x} = \mathsf{b}$ is $\begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$, so $\mathsf{b} = \begin{bmatrix} 2 \\ -3 \\ -6 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. (Many other solutions are possible.)

11. (a) [BB] Since $A\mathsf{x} = 0$, $(BA)\mathsf{x} = B(A\mathsf{x}) = B0 = 0$, so x is in the null space of BA.

(b) Since x is the column space of A, $\mathsf{x} = A\mathsf{y}$ for some vector y. Thus $\mathsf{x} = (BC)\mathsf{y} = B(C\mathsf{y})$ so x is in the column space of B.

4.2

1. (c) Any linear combination of $\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3$ will have third component 0. So v is **not** a linear combination of $\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3$.

2. (c) The question is whether there are scalars a, b and c so that $\mathsf{v} = a\mathsf{v}_1 + b\mathsf{v}_2 + c\mathsf{v}_3$. This is $A\mathsf{x} = \mathsf{v}$ with $A = \begin{bmatrix} 0 & 3 & 2 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \\ 2 & -1 & 1 \end{bmatrix}$ and $\mathsf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Gaussian elimination on the augmented matrix proceeds

$$\left[\begin{array}{ccc|c} 0 & 3 & 2 & 1 \\ -1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ 2 & -1 & 1 & 9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & 7 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -3 & -7 \\ 0 & 0 & 11 & 22 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -3 & -7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The system has the solution $c = 2$, $b = -1$, $a = 3$, so $\mathsf{v} = 3\mathsf{v}_1 - \mathsf{v}_2 + 2\mathsf{v}_3$ is in the span of v_1, v_2 and v_3.

(d) The question is whether there are scalars a, b and c so that $\mathsf{v} = a\mathsf{v}_1 + b\mathsf{v}_2 + c\mathsf{v}_3$. This is $A\mathsf{x} = \mathsf{v}$ with $A = \begin{bmatrix} 0 & 3 & 2 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \\ 2 & -1 & 1 \end{bmatrix}$ and $\mathsf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Gaussian elimination on the augmented matrix proceeds

$$\left[\begin{array}{ccc|c} 0 & 3 & 2 & 2 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 2 & -1 & 1 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 3 & -3 \end{array}\right].$$

The third equation reads $0 = 1$. There is no solution. Vector v is **not** in the span of the others.

4. (a) These vectors do not span R^3. The span of two nonparallel vectors in R^3 is a plane. To find the equation of this plane (not required, but a good student will find it anyway), we write $a\mathsf{u}_1 + b\mathsf{u}_2 = \begin{bmatrix} 2a+6b \\ a \\ -3a+b \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Solving the first equations for a and b gives $a = y$ and $x = 2a + 6b = 2y + 6b$, so $b = \frac{1}{6}(x - 2y)$. Then $z = -3a + b = -3y + \frac{1}{6}(x - 2y)$, so $x - 20y - 6z = 0$.

(b) These vectors do span R^3. Let $\mathsf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be any vector in R^3. We claim that there are scalars c_1, c_2 and c_3 so that $\mathsf{x} = c_1\mathsf{u}_1 + c_2\mathsf{u}_2 + c_3\mathsf{u}_3$. In matrix form, this is $\mathsf{x} = A\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, where $A = \begin{bmatrix} 2 & 6 & 1 \\ 1 & 0 & 1 \\ -3 & 1 & 1 \end{bmatrix}$. Since $\det A = -25 \neq 0$, A is invertible, so $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = A^{-1}\mathsf{x}$.

7. (c) U is not a subspace since it is not closed under addition: for example, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ are in U, but their sum $\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ is not because $4 \neq 1 + 1 + 1$.
(There are other possible answers. For instance, U is not closed under scalar multiplication. Also, U does not contain the zero vector.)

(e) This is not a subspace of R^3 because it is not closed under scalar multiplication: For example, $\mathsf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in U, but $\frac{1}{2}\mathsf{x}$ is not in U.

10. (b) The question is whether there are scalars c_1, c_2, c_3, not all 0, such that $c_1\mathsf{v}_1 + c_2\mathsf{v}_2 + c_3\mathsf{v}_3 = 0$. This is $A\mathsf{c} = 0$, with $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 4 \\ 3 & -1 & 11 \end{bmatrix}$ and $\mathsf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$. Gaussian elimination proceeds $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 4 \\ 3 & -1 & 11 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 6 \\ 0 & -7 & 14 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$.
There are many nontrivial solutions, for example, $c_1 = -3$, $c_2 = 2$, $c_3 = 1$ (hence $-3\mathsf{v}_1 + 2\mathsf{v}_2 + \mathsf{v}_3 = 0$), so the vectors are linearly dependent.

(c) The question is whether there are scalars c_1, c_2, c_3, not all 0, such that $c_1\mathsf{v}_1 + c_2\mathsf{v}_2 + c_3\mathsf{v}_3 = 0$. This is $A\mathsf{c} = 0$, with $A = \begin{bmatrix} -2 & 3 & 2 \\ 3 & 4 & 1 \\ 3 & 0 & 1 \\ 1 & 7 & 3 \end{bmatrix}$ and $\mathsf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$. Gaussian elimination proceeds $\begin{bmatrix} -2 & 3 & 2 \\ 3 & 4 & 1 \\ 3 & 0 & 1 \\ 1 & 7 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 7 & 3 \\ 0 & 17 & 8 \\ 0 & -17 & -8 \\ 0 & -21 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 7 & 3 \\ 0 & 17 & 8 \\ 0 & 21 & 8 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 7 & 3 \\ 0 & 1 & \frac{8}{17} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. The solution is $c_1 = c_2 = c_3 = 0$, so the vectors are linearly independent.