HW10

3.4

5. The eigenvalues of A are the roots of the polynomial $\det(A - \lambda I)$. Since $A - \lambda I$ is the triangular matrix $\begin{bmatrix} 5-\lambda & -2 & 6 \\ 0 & -1-\lambda & 9 \\ 0 & 0 & 3\lambda \end{bmatrix}$ and since the determinant of a triangular matrix is the product of its diagonal entries, $\det(A-\lambda I) = (5-\lambda)(-1-\lambda)(3-\lambda)$. The eigenvalues are 5, -1 and 3. Since the eigenvalues are distinct, A is diagonalizable by Theorem 3.4.14. There are six diagonal matrices to which A is similar, those whose diagonal entries are any of the six ways in which $5, -1, 3$ can be ordered.

7. We must determine whether or not A has 4 linearly independent eigenvectors. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is an eigenvector for $\lambda = -1$, then $\mathbf{x}$ is a solution to the homogeneous system $(A+I)\mathbf{x} = 0$. Gaussian elimination starting with $A + I$ proceeds

$$\begin{bmatrix} -1 & 3 & 1 & 0 \\ -1 & 3 & 1 & 0 \\ -4 & 3 & 4 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -9 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -1 & 0 \\ 0 & 1 & 0 & -\frac{1}{9} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so $x_3 = t$, $x_4 = 0$, $x_2 = 0$, $x_1 = t$ and $\mathbf{x} = t\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is an eigenvector for $\lambda = 2$, then $\mathbf{x}$ is a solution to the homogeneous system $(A - 2I)\mathbf{x} = 0$. Gaussian elimination starting with $A - 2I$ proceeds

$$\begin{bmatrix} -4 & 3 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -4 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 3 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so $x_4 = 0$, $x_3 = t$, $x_2 = t$, $x_1 = t$ and $\mathbf{x} = t\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$. We cannot find more than two linearly independent eigenvectors, so A is not diagonalizable.

10. (a) The characteristic polynomial of A is $\begin{vmatrix} 3-\lambda & 1 & 1 \\ -4 & -2-\lambda & -5 \\ 2 & 2 & 5-\lambda \end{vmatrix}$

$$= (3-\lambda)[(-2-\lambda)(5-\lambda)+10)] + 4[(5-\lambda)-2] + 2[-5-(-2-\lambda)]$$
$$= -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = (1-\lambda)(\lambda-2)(\lambda-3).$$

(b) Since the 3×3 matrix A has three different eigenvalues, 1, 2, and 3, A is diagonalizable by Theorem 3.4.14.

(c) The desired matrix P is a matrix whose columns are eigenvectors corresponding to 1, 2 and 3, **in that order**. To find the eigenspace for $\lambda = 1$, we solve $A - \lambda \mathrm{x} = 0$ with $\lambda = 1$. Gaussian elimination proceeds

$$\begin{bmatrix} 2 & 1 & 1 \\ -4 & -3 & -5 \\ 2 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ -4 & -3 & -5 \\ 2 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$ With $\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, we have $x_3 = t$ free, $x_2 = -3x_3 = -3t$, and $x_1 = -x_2 - 2x_3 = t$, so $\mathrm{x} = t\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$. To find the eigenspace for $\lambda = 2$, we solve $A - \lambda \mathrm{x} = 0$ with $\lambda = -1$.

Gaussian elimination proceeds $\begin{bmatrix} 1 & 1 & 1 \\ -4 & -4 & -5 \\ 2 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

With $\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, we have $x_2 = t$ free, $x_3 = 0$, and $x_1 = -x_2 - x_3 = -t$, so $\mathrm{x} = t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

To find the eigenspace for $\lambda = 3$, we solve $A - \lambda \mathrm{x} = 0$ with $\lambda = 3$.

Gaussian elimination proceeds $\begin{bmatrix} 0 & 1 & 1 \\ -4 & -5 & -5 \\ 2 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ -4 & -5 & -5 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

With $\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, we have $x_3 = t$ free, $x_2 = -x_3 = -t$, and $x_1 = -x_2 - x_3 = 0$, so $\mathrm{x} = t\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. We obtain $P = \begin{bmatrix} 1 & -1 & 0 \\ -3 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$.

14. (b) The characteristic polynomial of A is $\begin{vmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = -\lambda(2-\lambda) + 1 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$. There is only one eigenvalue, $\lambda = 1$, so if if A were diagonalizable, say $P^{-1}AP = D$, then $D = I$. But $P^{-1}AP = I$ implies $A = I$ which is not true. So A is not diagonalizable.

(c) The characteristic polynomial of A is $\begin{vmatrix} 5-\lambda & -2 \\ 1 & 2-\lambda \end{vmatrix} = (5-\lambda)(2-\lambda) + 2 = \lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4)$. Since A has two distinct eigenvalues, $\lambda = 3$ and $\lambda = 4$, it is diagonalizable. For $\lambda = 3$, a corresponding eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and, for $\lambda = 4$, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Let $P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$. Then $P^{-1}AP = D = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$.

14 (i) The characteristic polynomial of A is

$$\begin{vmatrix} 3-\lambda & 0 & 6 \\ 0 & -3-\lambda & 0 \\ 5 & 0 & 2-\lambda \end{vmatrix} = (-3-\lambda)\begin{vmatrix} 3-\lambda & 6 \\ 5 & 2-\lambda \end{vmatrix}$$
$$= (-3-\lambda)(\lambda^2 - 5\lambda - 24) = (-3-\lambda)(\lambda-8)(\lambda+3),$$

so A has eigenvalues $\lambda = -3$ and $\lambda = 8$. The eigenvectors for $\lambda = -3$ are found by solving $(A - \lambda I)\mathrm{x} = 0$ with $\lambda = -3$. Gaussian elimination proceeds $\begin{bmatrix} 6 & 0 & 6 \\ 0 & 0 & 0 \\ 5 & 0 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_2 = t$ and $x_3 = s$ is free and $x_1 = -x_3 = -s$. Eigenvectors are of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. To find the eigenvectors for $\lambda = 8$ we solve $(A - \lambda I)\mathrm{x} = 0$ with $\lambda = 8$. This time, Gaussian elimination proceeds $\begin{bmatrix} -5 & 0 & 6 \\ 0 & -11 & 0 \\ 5 & 0 & -6 \end{bmatrix} \to \begin{bmatrix} 5 & 0 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_3 = t$ is free, $x_2 = 0$, and $x_1 = \frac{6}{5}x_3 = \frac{6}{5}t$. Eigenvectors are of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{6}{5}t \\ 0 \\ t \end{bmatrix} = t'\begin{bmatrix} 6 \\ 0 \\ 5 \end{bmatrix}$. Two obvious eigenvectors for $\lambda = -3$ are $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and one for $\lambda = 8$ is $\begin{bmatrix} 6 \\ 0 \\ 5 \end{bmatrix}$. These three eigenvectors are linearly independent, so A is diagonalizable. The matrix $P = \begin{bmatrix} -1 & 0 & 6 \\ 0 & 1 & 0 \\ 1 & 0 & 5 \end{bmatrix}$ whose columns are the three indicated eigenvectors is invertible and $P^{-1}AP = D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 8 \end{bmatrix}$.