Math 2210 Final Exam

Name: _______________________

May 11, 2015

☐ Discussion Section 201 (M 1:25-2:15, Pueschel)
☐ Discussion Section 202 (M 2:30-3:20, Pueschel)
☐ Discussion Section 203 (M 2:30-3:20, Patotski)
☐ Discussion Section 204 (M 3:35-4:25, Patotski)
☐ Discussion Section 205 (M 3:35-4:25, Pueschel)

INSTRUCTIONS — READ THIS NOW

• Relax. Take a deep breath.
• Print your first and last name and check the box indicating which section you are in right now.
• This test has 9 problems on 9 pages (including the cover sheet and a page for scrap work). Look over your test package as soon as the exam begins. If you find any missing pages please ask a proctor for another test booklet.
• SHOW YOUR WORK. To receive full credit, your answers must be neatly written and logically organized. If you need more space, write on the back side of the preceding sheet, but be sure to label your work clearly.
• This is a 150 minute exam. You may leave early, but if you finish within the last 15 minutes, please stay in your seat. When time is called, put your pencil down immediately and pass your exam booklet to the aisle.
• This is a closed book exam. You are NOT allowed to use a calculator. Cell phones may NOT be used in the exam rooms, not even as time-keeping devices. All other aids are prohibited.
• Academic integrity is expected of all students of Cornell University at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination.

Please sign below to indicate that you have read and agree to these instructions.

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Signature of Student
1. (10 points) Let \( A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix} \)

(a) Construct a \( 4 \times 2 \) matrix \( B \) using only 1 and 0 as entries, such that \( AB = I_2 \).

(b) Is it possible that \( CA = I_4 \), for some \( 4 \times 2 \) matrix \( C \)? If yes find such a matrix \( C \), if not explain why it is impossible.

(a) (5 points) We use that the columns of \( AB \) are linear combinations of the columns of \( A \). By inspection we find that the matrix \( B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \) works.

(b) (5 points) Suppose that \( CA = I_4 \) for some \( 4 \times 2 \) matrix \( C \). Then \( C \) is onto because for any \( b \in \mathbb{R}^4 \), \( b = Cx \) where \( x = Ab \). However \( C \) has more rows than columns so it cannot be onto. The answer is that it is impossible.
2. (10 points) Let $V = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$, a subspace of $\mathbb{R}^3$. Let $A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{bmatrix}$.

(a) Find a basis $B$ of $V$. What is the dimension of $V$?

(b) Let $b = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$. Is $b \in V$? If yes find the coordinates of $b$ in the basis $B$.

(c) Solve the system $Ax = b$.

(a) (4 points) A basis of $V$ is given by the pivot columns of $A$. We row-reduce $A$:

$$A \sim \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & -6 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$  

The first two columns are pivot columns, so a basis of $V$ is

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}.$$  

The dimension of $V$ is the size of $B$ which is 2.

(b) (3 points) We look for weights $c_1, c_2$ such that

$$b = c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$  

The system is consistent, therefore $b \in V$. We find $c_1 = -1$ and $c_2 = 1$ which are the $B$-coordinates of $x$.

(c) (3 points) We row-reduce the augmented matrix:

$$\begin{bmatrix} 1 & 4 & 2 & 3 \\ 2 & 5 & 1 & 3 \\ 3 & 6 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & -3 & -3 & -3 \\ 0 & -6 & -6 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & -3 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$  

We find the reduced echelon form:

$$\sim \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$  

The third variable $x_3$ is free, so the solutions are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + 2x_3 \\ 1 - x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$  

This is a parametric equation of a line in $\mathbb{R}^3$. 

3. (10 points) Let \( A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \) and \( b = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \).

(a) Is the system \( Ax = b \) consistent? Justify your answer.
(b) Find a least square solution \( \hat{x} \) of \( Ax = b \).
(c) What is the distance from \( b \) to the column space of \( A \)?

(a) (2 points) We row-reduce the augmented matrix:

\[
\begin{bmatrix}
1 & 1 & 2 \\
1 & 2 & 2 \\
1 & 3 & 4
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & 2 \\
0 & 1 & 0 \\
0 & 2 & 2
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & 2 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{bmatrix}
\]

The system is inconsistent because the last row has only zeros and the augmented entry is non-zero.

(b) (4 points) We solve the normal equation \( A^T Ax = A^T b \). We find

\[
A^T A = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}
A^T b = \begin{bmatrix} 8 \\ 18 \end{bmatrix}
\]

We row-reduce the augmented matrix:

\[
\begin{bmatrix}
3 & 6 & 8 \\
6 & 14 & 18
\end{bmatrix}
\sim
\begin{bmatrix}
3 & 6 & 8 \\
0 & 1 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
3 & 0 & 2 \\
0 & 1 & 1
\end{bmatrix}
\]

The system has a unique solution \( \hat{x} = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} \).

(c) (4 points) The orthogonal projection of \( b \) to the column space of \( A \) is the vector \( A\hat{x} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 \\ 8 \\ 11 \end{bmatrix} \).

The distance from \( b \) to the column space is

\[
\|b - A\hat{x}\| = \left\| \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\| = \frac{1}{3} \sqrt{1^2 + 2^2 + 1^2} = \frac{1}{3} \sqrt{6} = \sqrt{\frac{2}{3}}
\]
Let \( Q(x) = -5x_1^2 + 2x_1x_2 - 5x_2^2 \).

(a) Find the matrix of the quadratic form \( Q(x) \).

(b) Determine if \( Q(x) \) is positive definite, positive semidefinite, negative definite, negative semidefinite or indefinite.

(c) Find the principal axes of \( Q \).

(d) What can you say about the sets \( C_1 = \{ x \in \mathbb{R}^2 : Q(x) = 1 \} \) and \( C_0 = \{ x \in \mathbb{R}^2 : Q(x) = 0 \} \).

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(a) \( 2 \) points \( Q(x) = x^T Ax \) for \( A = \begin{bmatrix} -5 & 1 \\ 1 & -5 \end{bmatrix} \).

(b) \( 5 \) points The characteristic equation of \( A \) is given by \( \det(A - \lambda I) = (\lambda + 4)(\lambda + 6) = 0 \).

Hence \( A \) has eigenvalues \( \lambda_1 = -4 \) and \( \lambda_2 = -6 \). Since all eigenvalues are \( < 0 \) the quadratic form is negative definite.

(c) \( 5 \) points The principal axes of \( Q \) are given by the normalized eigenvectors \( v_1 \) and \( v_2 \) of \( A \).

\[
A - \lambda_1 I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}
\]

\[
A - \lambda_2 I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}
\]

Hence \( v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) and \( v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).

(d) \( 3 \) points \( Q \) is negative definite, hence \( Q(x) < 0 \) for all \( x \neq 0 \). It follows that

\[ C_1 = \{ x \in \mathbb{R}^2 : Q(x) = 1 \} = \emptyset \quad \text{and} \quad C_0 = \{ x \in \mathbb{R}^2 : Q(x) = 0 \} = \{ 0 \} \.]
5. \textit{(10 points)} Solve the following linear system using Cramer’s rule:

\[
\begin{bmatrix}
2 & 2 & 0 \\
0 & -1 & -2 \\
3 & 0 & -7
\end{bmatrix}
x = \begin{bmatrix}
8 \\
-4 \\
-2
\end{bmatrix}.
\]

We solve \(Ax = b\) for \(A = \begin{bmatrix}
2 & 2 & 0 \\
0 & -1 & -2 \\
3 & 0 & -7
\end{bmatrix}\) and \(b = \begin{bmatrix}
8 \\
-4 \\
-2
\end{bmatrix}\).

\[
\det A = 2
\]

\[
det A_1(b) = \begin{vmatrix}
8 & 2 & 0 \\
-4 & -1 & -2 \\
-2 & 0 & -7
\end{vmatrix} = 8
\]

\[
det A_2(b) = \begin{vmatrix}
2 & 8 & 0 \\
0 & -4 & -2 \\
3 & -2 & -7
\end{vmatrix} = 0
\]

\[
det A_3(b) = \begin{vmatrix}
2 & 2 & 8 \\
0 & -1 & -4 \\
3 & 0 & -2
\end{vmatrix} = 4
\]

Hence \(x_1 = \frac{det A_1(b)}{det A} = 4\), \(x_2 = \frac{det A_2(b)}{det A} = 0\) and \(x_3 = \frac{det A_3(b)}{det A} = 2\).
6. *(10 points)* Compute the determinant:

\[
\begin{vmatrix}
3 & 4 & 4 & 4 \\
4 & 3 & 4 & 4 \\
4 & 4 & 3 & 4 \\
4 & 4 & 4 & 3 \\
\end{vmatrix}
\]

Subtraction of row \(i\) from row \(i+1\) for all \(i = 1, \ldots, 3\) gives

\[
\begin{vmatrix}
3 & 4 & 4 & 4 \\
4 & 3 & 4 & 4 \\
4 & 4 & 3 & 4 \\
4 & 4 & 4 & 3 \\
\end{vmatrix} = \begin{vmatrix}
3 \quad 4 \quad 4 \quad 4 \\
1 \quad -1 \quad 0 \quad 0 \\
0 \quad 1 \quad -1 \quad 0 \\
0 \quad 0 \quad 1 \quad -1 \\
\end{vmatrix}.
\]

Now adding column \(j+1\) to column \(j\) for all \(j = 3, \ldots, 1\) this is further equal to

\[
\begin{vmatrix}
3 & 4 & 8 & 4 \\
1 \quad -1 \quad 0 \quad 0 \\
0 \quad 1 \quad -1 \quad 0 \\
0 \quad 0 \quad 0 \quad -1 \\
\end{vmatrix} = \begin{vmatrix}
3 \quad 12 & 8 & 4 \\
1 \quad -1 \quad 0 \quad 0 \\
0 \quad 0 \quad -1 \quad 0 \\
0 \quad 0 \quad 0 \quad -1 \\
\end{vmatrix} = -15.
\]
7. (15 points) Let \( A = \begin{bmatrix} 3 & -2 & -2 \\ -4 & 1 & 2 \\ 8 & -4 & -5 \end{bmatrix} \).

(a) Check that \( v = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \) is an eigenvector of the matrix \( A \).

(b) Diagonalize the matrix \( A \). That is, find a diagonal matrix \( D \) and an invertible matrix \( P \), such that \( A = PDP^{-1} \). You don’t need to compute \( P^{-1} \).

(c) Compute \( A^n \), for all \( n \geq 0 \).

(a) (2 points) \( Av = v \), hence \( v \) is eigenvector corresponding to the eigenvalue 1.

(b) (8 points) We compute the characteristic equation

\[
\text{det}(A - \lambda I) = \begin{vmatrix} 3 - \lambda & -2 & -2 \\ -4 & 1 - \lambda & 2 \\ 8 & -4 & -5 - \lambda \end{vmatrix} = \lambda^3 + \lambda^2 - \lambda - 1 = (\lambda - 1)(\lambda + 1)^2 = 0.
\]

A has eigenvalues \( \lambda_1 = 1 \) with multiplicity 1 and \( \lambda_2 = -1 \) with multiplicity 2.

\( (A + I) = \begin{bmatrix} 4 & -2 & -2 \\ -4 & 2 & 2 \\ 8 & -4 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \)

This gives two independent eigenvectors \( v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \) and \( v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \).

Hence \( A = PDP^{-1} \) for \( P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \) and \( D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \).

(c) (5 points) \( A^n = PD^n P^{-1} = \begin{bmatrix} 1^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & (-1)^n \end{bmatrix} \).

If follows that

\[
A^n = \begin{cases} A & \text{if } n \text{ odd} \\ I & \text{if } n \text{ even.} \end{cases}
\]
8. (10 points) Let \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \) be vectors in \( \mathbb{R}^4 \).

(a) Assume the set \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \} \) is linearly independent. Is then the set \( \{ \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_3 + \mathbf{v}_4 \} \) also linearly independent? Give a proof or a counterexample.

(b) Assume the set \( \{ \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_3 + \mathbf{v}_4 \} \) is linearly independent. Is then the set \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \} \) also linearly independent? Give a proof or a counterexample.

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(a) (5 points) Assume \( \mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2 \) and \( \mathbf{w}_2 = \mathbf{v}_3 + \mathbf{v}_4 \) are linearly dependent, Hence there exists a linear dependence relation \( c_1 \mathbf{w}_1 + c_2 \mathbf{w}_2 = \mathbf{0} \) with weights \( c_1, c_2 \) not both equal to zero. Hence \( c_1 \mathbf{v}_1 + c_1 \mathbf{v}_2 + c_2 \mathbf{v}_3 + c_2 \mathbf{v}_4 = \mathbf{0} \) is a linear dependence relation for \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \).

This proves that the statement is true.

(b) (5 points) The statement is false. Let \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \) and \( \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \).

Hence \( \mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \) and \( \mathbf{w}_2 = \mathbf{v}_3 + \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \) are linearly independent. But the set \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \) is linearly dependent since it contains the zero vector.
9. (10 points) Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a basis for $\mathbb{R}^4$.

(a) Find a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ that maps $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ onto $\text{Span}\{\mathbf{v}_3, \mathbf{v}_4\}$.

(b) Find the matrix $[T]_B$, that is the matrix for $T$ relative to the basis $B$.

(a) (5 points) The linear map $T$ with $T(\mathbf{v}_1) = \mathbf{v}_3$, $T(\mathbf{v}_2) = \mathbf{v}_4$, $T(\mathbf{v}_3) = \mathbf{v}_1$ and $T(\mathbf{v}_4) = \mathbf{v}_2$ has the desired property.

(b) (5 points)

$$[T]_B = \begin{bmatrix} T(\mathbf{v}_1) & T(\mathbf{v}_2) & T(\mathbf{v}_3) & T(\mathbf{v}_4) \end{bmatrix}_B$$

$$= \begin{bmatrix} [\mathbf{v}_3]_B & [\mathbf{v}_4]_B & [\mathbf{v}_1]_B & [\mathbf{v}_2]_B \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$
Common Mistakes:

1.