Math 2210 Final Exam  
Name: _______________________

December 14, 2015

Discussion 1 (12:20-1:10, Frederik De Keersmaeker)
Discussion 2 (1:25-2:15, Benjamin Hoffman)
Discussion 3 (1:25-2:15, Frederik De Keersmaeker)
Discussion 4 (1:25-2:15, Ian Lizarraga)
Discussion 5 (2:30-3:20, Benjamin Hoffman)
Discussion 6 (2:30-3:20, Frederik De Keersmaeker)
Discussion 7 (3:35-4:25, Benjamin Hoffman)
Discussion 8 (3:35-4:25, Ian Lizarraga)
Discussion 9 (7:30-8:20, Ian Lizarraga)

INSTRUCTIONS — READ THIS NOW

• Relax. Take a deep breath.
• Print your first and last name and check the box indicating which section you are in right now.
• This test has 10 problems on 12 pages (including the cover sheet and a page for scratch work). Look over your test package as soon as the exam begins. If you find any missing pages please ask a proctor for another test booklet.
• SHOW YOUR WORK. To receive full credit, your answers must be neatly written and logically organized. If you need more space, write on the back side of the preceding sheet, but be sure to label your work clearly.
• This is a 150 minute exam. You may leave early, but if you finish within the last 15 minutes, please stay in your seat. When time is called, put your pencil down immediately and pass your exam booklet to the aisle.
• This is a closed book exam. You are NOT allowed to use a calculator. Cell phones may NOT be used in the exam rooms, not even as time-keeping devices. All other aids are prohibited.
• Academic integrity is expected of all students of Cornell University at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination.

Please sign below to indicate that you have read and agree to these instructions.

______________________________
Signature of Student
1. (10 points) Consider the $4 \times 4$ matrix

$$A = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 3 & 1 \\
1 & 3 & 2 & -2 \\
1 & 1 & -2 & -2
\end{bmatrix}. $$

(a) Find a basis for the column space of $A$.
(b) Find a basis for the null space of $A$.
(c) The vectors in the basis you found in (a) should be perpendicular to the vectors in the basis you found in (b). Tell us an abstract theorem that would have let you conclude those two bases were orthogonal to each other without doing any computations.

(a) (4 points) Row-reducing we can find that $A$ is row-equivalent to the REF matrix

$$A = \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 2 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}. $$

This has pivots in the first three columns, so a basis for the column space may be taken to be the first three columns of $A$:

$$\begin{bmatrix}1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix}1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix}1 \\ 1 \\ 2 \end{bmatrix}. $$

(b) (4 points) Further row-reducing we get that $A$ has RREF form

$$A = \begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & -2 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix};$$

from this we can read off from the one non-pivot column that the null space is spanned by the vector

$$\begin{bmatrix}2 \\ -2 \\ -1 \\ 1 \end{bmatrix}. $$

(c) (2 points) This follows from the theorem that $\text{Col}(A)^\perp = \text{Nul}(A^T)$, since this matrix is symmetric so satisfies $A = A^T$. Alternatively, it follows from the spectral theorem - the null space is the eigenspace for zero, which is orthogonal to the nonzero eigenspaces (which span the column space).
2. (15 points)

(a) Compute the characteristic polynomial for the following matrix, and find its eigenvalues in \( \mathbb{C} \). Does it have any eigenvectors in \( \mathbb{R}^2 \)?

\[
A = \begin{bmatrix}
-2 & 5/2 \\
-2 & 2
\end{bmatrix}.
\]

(b) Find matrices \( P, C \) such that \( A = PCP^{-1} \), where \( P \) is an invertible 2 \( \times \) 2 matrix and \( C \) is of the form

\[
C = \begin{bmatrix}
a & -b \\
b & a
\end{bmatrix}.
\]

(c) The matrix \( C \) defines a familiar transformation \( \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) - what angle is it a rotation by? Then describe the linear transformation \( \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) determined by \( A \) as a “rotation” along some sort of shape, and write down two vectors \( \mathbf{v}_1, \mathbf{v}_2 \) such that \( A \) “rotates” one to the other. Sketch the shape, and include your two vectors.

**Hint:** Search for two vectors that satisfy \( A\mathbf{v}_1 = \mathbf{v}_2 \) and \( A\mathbf{v}_2 = -\mathbf{v}_1 \).

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(a) (6 points) The characteristic polynomial is \( \lambda^2 + 1 = 0 \), which has eigenvalues \( \pm i \). This has no real eigenvalues, so no real eigenvectors.

(b) (6 points) By results in the book, we get \( C \) by writing the eigenvalues as \( a \pm bi \), and then \( P = [\Re(v) \ \Im(v)] \) for \( v \) an eigenvector for \( a - bi \). Since our eigenvalues are \( 0 \pm i \) we get

\[
C = \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix};
\]

then the eigenvectors for \( A \) for \( \lambda = -i \) are solutions to

\[
\begin{bmatrix}
-2 + i & 5/2 \\
-2 & 2 + i
\end{bmatrix} v = 0,
\]

and we can take one such solution to be

\[
v = \begin{bmatrix}
5/2 \\
2 - i
\end{bmatrix},
\]

and thus

\[
P = [\Re(v) \ \Im(v)] = \begin{bmatrix}
5/2 & 0 \\
2 & -1
\end{bmatrix}.
\]

(c) (3 points) The matrix \( C \) is a rotation by \( \pi/2 \) (i.e. 90 degrees). Since \( A \) is obtained by a change-of-basis from \( C \), this means \( A \) “rotates” the first column of \( P \) to the second. So \( A \) can be thought of as a “rotation” along an ellipse determined by these vectors.
3. \textit{(15 points)} Consider the $4 \times 3$ matrix
\[
A = \begin{bmatrix}
1 & 3 & 2 \\
1 & -1 & 0 \\
1 & 3 & 0 \\
1 & -1 & -2
\end{bmatrix}.
\]

(a) Use the Gram-Schmidt process to find an orthogonal basis for the column space of $A$.
(b) Normalize your basis from part (a) to obtain an orthonormal basis for the column space of $A$.
(c) Find a $QR$-decomposition for the matrix $A$.
(d) Find the least-squares solution $\hat{x}$ to the equation $Ax = b$ for
\[
b = \begin{bmatrix}
4 \\
0 \\
0 \\
4
\end{bmatrix}.
\]

(a) (5 points) The basis you obtain is (up to positive scalar multiples, perhaps)\[
\begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix}, \begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix}, \begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix}.
\]

(b) (2 points) Normalizing gives\[
\begin{bmatrix}
1/2 \\
1/2 \\
1/2 \\
1/2
\end{bmatrix}, \begin{bmatrix}
1/2 \\
-1/2 \\
1/2 \\
-1/2
\end{bmatrix}, \begin{bmatrix}
1/2 \\
1/2 \\
-1/2 \\
-1/2
\end{bmatrix}.
\]

(c) (3 points) We can take the matrix $Q$ to have the columns determined in part (b), and then compute $R$ as $Q^T A$ to get:\[
Q = \begin{bmatrix}
1/2 & 1/2 & 1/2 \\
1/2 & -1/2 & 1/2 \\
1/2 & 1/2 & -1/2 \\
1/2 & -1/2 & -1/2
\end{bmatrix}, \quad R = \begin{bmatrix}
2 & 2 & 0 \\
0 & 4 & 2 \\
0 & 0 & 2
\end{bmatrix}.
\]

(d) (5 points) The least-squares solution is the solution $\hat{x}$ to $A^T A \hat{x} = A^T b$, or equivalently to $R \hat{x} = Q^T b$. Solving this latter equation we can find\[
\hat{x} = \begin{bmatrix}
2 \\
0
\end{bmatrix}.
\]
4. (15 points) For $a, b, c \in \mathbb{R}$, consider the $4 \times 4$ determinant

$$f(a, b, c) = \begin{vmatrix} 0 & a^2 & b^2 & 1 \\ a^2 & 0 & c^2 & 1 \\ b^2 & c^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}.$$ 

(a) Show that $f(a, 0, 0) = a^4$ for every $a \in \mathbb{R}$.
(b) Show that $f(a, b, 0) = (a + b)^2(a - b)^2$ for every $a, b \in \mathbb{R}$.
(c) Using only swap operations on both rows and columns, show that $f(b, a, c) = f(a, b, c)$ for every $a, b, c \in \mathbb{R}$.

(difficult question worth 5 points of extra credit) Show that $f(a, b, c) = (a + b + c)(a - b - c)(b - a - c)(c - a - b)$ for every $a, b, c \in \mathbb{R}$.

(a) (5 points) We expand the third column and then the third row:

$$f(a, 0, 0) = \begin{vmatrix} 0 & a^2 & 0 & 1 \\ a^2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & a^2 & 1 \\ a^2 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} 0 & a^2 \\ a^2 & 0 \\ 0 & 0 \end{vmatrix} = a^4.$$

(b) (5 points) We subtract the third row from the second row and then expand the second row:

$$f(a, b, 0) = \begin{vmatrix} 0 & a^2 & b^2 & 1 \\ a^2 & 0 & 0 & 1 \\ b^2 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & a^2 & b^2 & 1 \\ a^2 - b^2 & 0 & 0 & 0 \\ b^2 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = -(a^2 - b^2)^2 \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -(a^2 - b^2)\cdot(a^2 - b^2) = (a^2 - b^2)^2.$$

Thus $f(a, b, 0) = (a^2 - b^2)^2 = (a + b)^2(a - b)^2$.

(c) (5 points) We permute the second and third row, and then the second and third columns:

$$f(a, b, c) = \begin{vmatrix} 0 & a^2 & b^2 & 1 \\ b^2 & 0 & c^2 & 1 \\ a^2 & 0 & c^2 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & a^2 & b^2 & 1 \\ b^2 & 0 & c^2 & 1 \\ a^2 & c^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = f(b, a, c).$$

(extra 5 points credit question) We subtract the second row from the first row and the third row and then expand the fourth column:

$$f(a, b, c) = \begin{vmatrix} 0 & a^2 & b^2 & 1 \\ a^2 & 0 & c^2 & 1 \\ b^2 & c^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & a^2 & b^2 - c^2 & 0 \\ a^2 & 0 & c^2 & 1 \\ b^2 - a^2 & c^2 & -c^2 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -a^2 & a^2 & b^2 - c^2 & 0 \\ b^2 - a^2 & c^2 & -c^2 & 0 \\ 0 & -2a^2 & b^2 - c^2 - a^2 & -2c^2 \\ 1 & 0 & 0 \end{vmatrix}$$

we then subtract the second column from the first and the third column and expand the third row:

$$= \begin{vmatrix} -2a^2 & a^2 & b^2 - c^2 - a^2 & 0 \\ b^2 - a^2 & c^2 & -2c^2 & 0 \\ 0 & -2a^2 & b^2 - c^2 - a^2 & -2c^2 \\ 1 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} -2a^2 & b^2 - c^2 - a^2 & 0 \\ b^2 - a^2 & c^2 & -c^2 & 0 \\ 0 & -2a^2 & b^2 - c^2 - a^2 & -2c^2 \\ 1 & 0 & 0 \end{vmatrix} = (b^2 - a^2 - c^2)^2 - 4a^2c^2$$

we then factor the polynomial:

$$= (b^2 - a^2 - c^2 - 2ac)(b^2 - a^2 - c^2 + 2ac) = (b^2 - (a + c)^2)(b^2 - (a - c)^2)$$

$$= (b + (a + c))(b - (a + c))(b - (a - c))(b - (a - c))$$

$$= (a + b + c)(a - b - c)(b - a - c)(c - a - b).$$
5. (20 points) Let \( x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \) and \( z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \) be three vectors in \( \mathbb{R}^2 \).

(a) Let \( A = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{bmatrix} \). Show that \( \det(A)^2 = \begin{vmatrix} x \cdot x + 1 & x \cdot y + 1 & x \cdot z + 1 \\ x \cdot y + 1 & y \cdot y + 1 & y \cdot z + 1 \\ x \cdot z + 1 & y \cdot z + 1 & z \cdot z + 1 \end{vmatrix} \).

(b) Show that
\[
\begin{vmatrix} x \cdot x & x \cdot y & x \cdot z \\ x \cdot y & y \cdot y & y \cdot z \\ x \cdot z & y \cdot z & z \cdot z \end{vmatrix} = -\begin{vmatrix} x \cdot x & x \cdot y & x \cdot z \\ y \cdot y & y \cdot y & y \cdot z \\ -1 \end{vmatrix}.
\]

Hint: Begin by adding the fourth Row to the first, second and third Rows.

(c) Justify carefully that
\[
\begin{vmatrix} x \cdot x & x \cdot y & x \cdot z \\ x \cdot y & y \cdot y & y \cdot z \\ x \cdot z & y \cdot z & z \cdot z \end{vmatrix} = 0. \quad \text{Hint: Compute } B^T B \text{ where } B = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Remark: Combining the above we can deduce that
\[
4 \det(A)^2 = \begin{vmatrix} 0 & ||x - y||^2 & ||x - z||^2 \\ ||x - y||^2 & 0 & ||y - z||^2 \\ ||x - z||^2 & ||y - z||^2 & 1 \end{vmatrix},
\]

Using the computation in Problem 4., we have given a proof of Héron’s formula
\[
\text{Area}(ABC) = \frac{1}{4} \sqrt{(a + b + c)(a + b - c)(a + c - b)(b + c - a)}
\]

for the Area of a triangle with side lengths \( a, b, c \).

(a) (5 points) We compute that
\[
A^T A = \begin{bmatrix} x \cdot x + 1 & x \cdot y + 1 & x \cdot z + 1 \\ x \cdot y + 1 & y \cdot y + 1 & y \cdot z + 1 \\ x \cdot z + 1 & y \cdot z + 1 & z \cdot z + 1 \end{bmatrix}.
\]

Furthermore \( \det(A^T A) = \det(A^T) \det(A) = \det(A) \det(A) = \det(A)^2 \).

(b) (5 points) Adding the fourth row to the first, second and third rows, the \( 4 \times 4 \) determinant is equal to
\[
\begin{vmatrix} x \cdot x + 1 & x \cdot y + 1 & x \cdot z + 1 & 0 \\ x \cdot y + 1 & y \cdot y + 1 & y \cdot z + 1 & 0 \\ x \cdot z + 1 & y \cdot z + 1 & z \cdot z + 1 & 0 \\ 1 & 1 & 1 & -1 \end{vmatrix}.
\]

It remains to expand the fourth column.

(c) (6 points) We have
\[
\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}. \quad \text{By the multilinearity of the determinant, we deduce that}
\]
\[
\begin{vmatrix} x \cdot x & x \cdot y & x \cdot z \\ x \cdot y & y \cdot y & y \cdot z \\ x \cdot z & y \cdot z & z \cdot z \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x \cdot x & x \cdot y & x \cdot z \\ x \cdot y & y \cdot y & y \cdot z \\ x \cdot z & y \cdot z & z \cdot z \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} x \cdot x & x \cdot y & x \cdot z \\ x \cdot y & y \cdot y & y \cdot z \\ x \cdot z & y \cdot z & z \cdot z \\ 1 & 1 & 1 \end{vmatrix}.
\]

It remains to expand the last determinant on the right-hand side with respect to the fourth column.

(d) (4 points) The computation is similar to (a), and it suffices to observe that \( \det(B) = 0 \).
Let $\mathbf{u}$ be a nonzero vector in $\mathbb{R}^n$, denote by $A_\mathbf{u}$ the $n \times n$ matrix $\mathbf{uu}^T$.

(a) Find the null space and the column space of $A_\mathbf{u}$;
(b) Find the eigenvalues of the matrix $A_\mathbf{u}$ and describe the eigenspaces;
(c) Use the previous part to find the eigenvalues and eigenvectors of the matrix $B = \text{Id} + A_\mathbf{u}$;
(d) Find the determinant of the matrix $B$ — this should be expressed in terms of the vector $\mathbf{u}$;
(e) Use the previous part to compute the determinant of the matrix
\[
\begin{bmatrix}
10 & 12 & 15 \\
12 & 17 & 20 \\
15 & 20 & 26
\end{bmatrix}.
\]

**Hint:** In parts b,c) the eigenvectors are not unique; part e) can be done without multiplying two digit numbers.

(a) (4 points) The column space of the matrix is 1 dimensional and is spanned by the vector $\mathbf{u}$. The null space has dimension $n - 1$ and consists of all vectors $\mathbf{w}$ such that $\mathbf{u}^T \mathbf{w} = 0$.

(b) (4 points) 0 is an eigenvalue with multiplicity $n - 1$ and the corresponding eigenspace is the null space of the matrix $A$. The other eigenvector is $\mathbf{u}$ and the corresponding eigenvalue is $\mathbf{u}^T \mathbf{u}$.

(c) (4 points) Since $B = \text{Id} + A_\mathbf{u}$ the eigenvectors of $A$ are also eigenvectors of $B$ but the eigenvalues are increased by 1. Any vector in the null space of $A$ is an eigenvector of $B$ with eigenvalue 1; thus 1 is an eigenvalue with multiplicity $n - 1$. The other eigenvector is $\mathbf{u}$ and the corresponding eigenvalue is $1 + \mathbf{u}^T \mathbf{u}$.

(d) (4 points) The determinant of $B$ is the product of the eigenvalues, therefore $\det B = 1^{n-1}(1 + \mathbf{u}^T \mathbf{u})$.

(e) (4 points) Notice that the matrix can be obtained starting from the vector $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$. Therefore the determinant is equal to $1^{3-1}(1 + 3^2 + 4^2 + 5^2) = 51$. 

\[
\begin{bmatrix}
10 & 12 & 15 \\
12 & 17 & 20 \\
15 & 20 & 26
\end{bmatrix}.
\]
7. (15 points)

Let \( A \) denote the \( 2 \times 2 \) matrix \[
\begin{bmatrix}
1 & 2 \\
0 & 2
\end{bmatrix}
\].

(a) Find the eigenvectors and the eigenvalues of the matrix \( A \).

(b) Consider the linear transformation \( T \) acting on the space \( \text{Mat}_{2 \times 2} \) of \( 2 \times 2 \) matrices defined by \( T(X) = AX \).

The space \( \text{Mat}_{2 \times 2} \) has a basis \( B = \{e_{11}, e_{12}, e_{21}, e_{22}\} \), where \( e_{ij} \) denote the elementary matrices with only one 1 and 3 zeros. Write the matrix \( M \) of the transformation \( T \) in this basis \( B \).

(c) Find the eigenvectors and the eigenvalues for the matrix \( M \).

---

(a) (4 points) The eigenvalues are 1 and 2 and the eigenvectors are \[
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\] and \[
\begin{bmatrix}
2 \\
1
\end{bmatrix}
\].

(b) (5 points) With respect to the given basis \( B \) the matrix is

\[
\begin{bmatrix}
1 & 0 & 2 & 0 \\
0 & 1 & 0 & 2 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 2
\end{bmatrix}
\]

(c) (6 points) Since the matrix is upper triangular the eigenvalues are just the diagonal entries, i.e., 1 and 2 with algebraic multiplicity 2. The geometric multiplicities are also 2 and the eigenvectors are

\[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
0 \\
2 \\
1
\end{bmatrix}.
\]
8. (15 points) Consider the transformation $T$ which sends a function $f : \mathbb{R} \to \mathbb{R}$ to the function $T(f)$ defined by $T(f)(x) = f(2x + 1)$.

(a) Show that $T$ is a linear transformation.

(b) The transformation $T$ takes a polynomial of degree at most $k$ to another polynomial of degree at most $k$. Thus, $T$ define a transformation from $\mathbb{P}_2$ to $\mathbb{P}_2$ (quadratic polynomials). Choose a basis $\mathcal{B}$ of $\mathbb{P}_2$ and determine the matrix of $T$ with respect to this basis.

(c) Find the eigenvalues and the eigenvectors of $T$ acting on $\mathbb{P}_2$.

(a) (5 points) Verify the axioms...

(b) (5 points) We have $T(1) = 1$; $T(x) = 2x + 1$ and $T(x^2) = (2x + 1)^2$, therefore the matrix of $T$ is $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$.

(c) (5 points) The matrix $M$ is upper triangular and the eigenvalues are just the diagonal entries. The eigenvalues of $M$ are 1, 2 and 4 and the eigenvectors are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Note that finding the eigenvectors requires solving a simple system of linear equations. The corresponding eigenvectors of the transformation $T$ are the functions $1$, $x + 1$ and $(x + 1)^2$. 
9. (15 points) Consider the following product on the space $\mathbb{P}_3$ of polynomials of degree at most 3 defined by

$$\langle f, g \rangle = f(0)g(0) + f(1)g(1) + f(-1)g(-1) + f'(0)g'(0).$$

Here $f(1)$ denotes the values of the polynomial $f$ at the point $x = 1$, also $f'$ denotes the derivative of $f$.

(a) Write down the axioms of an inner product, and verify all of them besides “positive definiteness” for this product.

(b) Show that $\langle f, f \rangle \geq 0$ for any $f \in \mathbb{P}_3$.

(c) Show that $\langle f, f \rangle > 0$ if $f \neq 0$.

(a) (5 points) Clearly the $\langle f, g \rangle$ is linear in $f$ and $g$ and symmetric.

(b) (4 points) It is easy to see that $\langle f, f \rangle$ is a sum of squares therefore in non-negative.

(c) (6 points) The only way $\langle f, f \rangle = 0$ is if $f$ is zero at 0, 1, $-1$ and $f'$ is zero at 0. However the only polynomials of degree 3 which have zeros at 0, 1, $-1$ are multiples of $x^3 - x$ and their derivatives does not vanish at 0 unless the polynomial is identically 0.

Note that the last part is not true if we replace $\mathbb{P}_3$ with $\mathbb{P}_4$, since $\langle x^4 - x^2, x^4 - x^2 \rangle = 0$. 
10. (10 points)

(a) Diagonalize the matrix

\[ B = \begin{bmatrix} 7 & 3 \\ -6 & -2 \end{bmatrix}. \]

(b) Find a diagonalizable 2 \times 2 matrix \( A \) such that \( A^2 = B \).

(a) (5 points) The eigenvalues for \( B \) are 4 and 1 and the corresponding eigenvectors are \( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \) and \( \begin{bmatrix} 1 \\ -2 \end{bmatrix} \). Thus,

\[ B = PDP^{-1} \text{ where} \]

\[ D = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}. \]

(b) (5 points) One can diagonalize \( B = PDP^{-1} \) and then take a square root of the matrix \( D \) by taking square root of each diagonal entry. Finally we can use \( A = P\sqrt{D}P^{-1} \).

An alternative way to do this is to see that the eigenvectors for \( A \) need to be also eigenvectors of \( B \) and their eigenvalues are related. The eigenvalues for \( B \) are 4 and 1 and the corresponding eigenvectors are \( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \) and \( \begin{bmatrix} 1 \\ -2 \end{bmatrix} \). Thus these needs to be the eigenvectors for \( A \) with eigenvalues \( \pm 2, \pm 1 \). Taking only the plus signs we get

\[ A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}. \]

Another way to solve this is to make \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) compute \( A^2 \) and obtained 4 quadratic equations in 4 variables. Solving the system is messy but doable (there are 4 solutions).