Math 2210 - Linear Algebra
Final exam - 13 December 2013 - 9 to 11:30am

Name and NetID: ________________________________

Whose discussion section are you enrolled in? Circle one.

Yash Lodha   Wai-kit Yeung

At what time is the discussion section you enrolled in? Circle one.

1:25-2:15pm  2:30-3:20pm  3:35-4:25pm

INSTRUCTIONS

• This test has 6 problems on 6 pages, worth a total of 100 points. Check if you have all 6 pages with questions.

• If you need more space than you are given under a question, clearly indicate where the remaining work is and point out where your final answer is. You also have a 2-sided page for scratchwork at the end.

• No books or electronic devices allowed. You are allowed a two-sided letter size paper of notes.

• Please show all your work and justify your answers.

Academic integrity is expected of all Cornell University students at all times, whether in the presence or absence of members of the faculty.

Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination.

Please sign below to indicate that you have read and agree to these instructions.

Signature of Student
1. (20 points)

Let $A = \begin{bmatrix} 0 & 2 & -1 & -7 \\ 0 & 0 & 5 & 0 \\ 0 & 4 & -2 & 6 \end{bmatrix}$. Find an orthonormal basis for each of the following vector spaces.

(a) Column space of $A$.
(b) Row space of $A$.
(c) Nullspace of $A$.
(d) The orthogonal complement of the column space of $A$. 
2. (15 points)

Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

(a) Find a singular value decomposition $A = U \Sigma V^T$.
(b) Using part (a) or otherwise, find unit vectors $u$ and $v$ and a positive real number $\sigma$ such that $A = \sigma u v^T$. 
3. (15 points)

Let $V$ be the six-dimensional vector space of functions $g : \mathbb{R}^2 \to \mathbb{R}$ of the form

$$g(x, y) = ax^2 + bxy + cy^2 + dx + ey + f.$$ 

Let $W$ be the vector space of polynomials in $x$ of degree at most 2.

(a) Write down a basis for $V$ and a basis for $W$.

(b) In the bases you picked in part (a), write down the standard matrix of the linear transformation $T : V \to W$ that takes $g(x, y)$ to $h(x) := g(x, x)$. 

4. (15 points)

Let \( q_1, q_2, q_3 \) be orthonormal column vectors and \( A = q_1 q_1^T + 2q_2 q_2^T + 3q_3 q_3^T \) a matrix.

(a) Show that \( A \) has eigenvalues 1, 2 and 3. What are the eigenvectors?

(b) Solve the initial value problem \( x'(t) = Ax(t) \) with \( x(0) = q_2 - q_3 \). (Your solution may be in terms of \( q_1, q_2, q_3 \).)

(c) Write down the general formula for the solutions of the differential equation \( y'(t) = By(t) \) with \( B = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \), and sketch their typical trajectories on \( \mathbb{R}^2 \).
5. (20 points)

Let $A$ be a real symmetric matrix with orthonormal columns.

(a) How is $A^{-1}$ related to $A$? (in addition to being its inverse)

(b) What number(s) can be eigenvalues of $A$? Why?

(c) Here is an example of such an $A$. What are its eigenvalues? (Hint: You can use part (b) to get this answer more quickly!)

\[ A = \begin{bmatrix}
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2}
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
1 & -1 & 1 & -1 \\
-1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 \\
-1 & -1 & 1 & 1
\end{bmatrix} \]

(d) Name the columns of the matrix $A$ from part (c), from left to right, $q_1$, $q_2$, $q_3$, $q_4$, and let

\[ B = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}. \]

Find the least squares solution $\hat{x}$ to the matrix equation $Bx = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. 

6. (15 points)

(a) If $A$ is a real symmetric matrix, which of the following are necessarily positive definite? $A^2$, $A^3$, $A + I$, $(A^2 + I)^{-1}$.

(b) Let $v$ be a column vector and $B = vv^T$. Then the matrix $B$ is: positive definite / negative definite / positive semidefinite / negative semidefinite / indefinite.

(c) Let $u$ and $v$ be orthonormal vectors, $b = 3u + v$ and $V = \text{span}\{u + v\}$.
   True or FALSE? The orthogonal projection of $b$ onto $V$ is $2u + 2v$. 

This page is for scratch work, it will not be graded unless you point us here from the page where the question was posed.

We hope that you enjoyed Math 2210 Linear Algebra, have a great break!