(20 points) Consider the matrix
\[ A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 6 & 8 & 16 \end{bmatrix}. \]

(a) Find a basis for the column space of \( A \).
(b) Find a basis for the null space of \( A \).
(c) Find a basis for the row space of \( A \).
(d) What is the rank of \( A \)? How does it relate to the dimension of the spaces described in (a), (b) and (c) above?

Solution

We begin by row reducing \( A \):
\[ A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 6 & 8 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}. \]

(a) Since the pivots appear in the 1st and 3rd columns, the corresponding columns of \( A \) form a basis for \( \text{Col}(A) \), so a basis is \( \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix} \right\} \).

(In this case, you could also argue that since \( \text{Col}(A) \) is a 2-dimensional subspace of \( \mathbb{R}^2 \), it must in fact be equal to \( \mathbb{R}^2 \), and then give any basis for \( \mathbb{R}^2 \), for example \( \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \).

(b) From the row reduced echelon form of \( A \) we see that for a vector \( x \) to be in the nullspace, it has to satisfy
\[
\begin{align*}
x_1 + 2x_2 &= 0 \\
x_3 + 2x_4 &= 0
\end{align*}
\]

which implies
\[
\begin{align*}
x_1 &= -2x_2 \\
x_3 &= -2x_4
\end{align*}
\]

and so
\[ x = \begin{bmatrix} -2x_2 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}. \]

A basis for \( \text{Nul}(A) \) is \( \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\} \).

(c) The nonzero rows of the row reduced echelon form of \( A \) form a basis for \( \text{Row}(A) \), so a basis is \( \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix} \right\} \).

(over)
(In this case, you could also argue that since the dimension of Row($A$) is equal to the dimension of Col($A$), which is 2, the two rows of $A$ must form a basis for Row($A$).)

(d) We have $\text{rank}(A) = \dim(\text{Col}(A)) = \dim(\text{Row}(A)) = 2$ and $\dim(\text{Nul}(A)) = (\text{number of columns of } A) - \text{rank}(A) = 4 - 2 = 2.$

2 (20 points). (a) Find a basis for the subspace of $\mathbb{P}_2$ spanned by the following polynomials:

$$p_1(t) = 1 + t + t^2$$
$$p_2(t) = 2 + t$$
$$p_3(t) = t + 2t^2.$$ 

(b) What are the coordinates of the polynomial $q(t) = 4 + 5t + 6t^2$ in the basis that you found in part (a)?

Solution

Fix a basis for $\mathbb{P}_2$ so that you can work with the coordinate vectors instead of with the polynomials: $B = \{1, t, t^2\}$.

(a) Using coordinate vectors, this question is equivalent to finding a basis for the subspace of $\mathbb{R}^3$ spanned by the vectors

$$[p_1(t)]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, [p_2(t)]_B = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, [p_3(t)]_B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$ 

A basis for the span of these vectors will be a basis for the column space of the matrix that has these vectors as columns:

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and so it will consist of the first and second columns (the ones where pivots show up) of the original matrix, $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$. Using the coordinate mapping to revert back to $\mathbb{P}_2$, this corresponds to $\{p_1(t), p_2(t)\}$ being a basis for the original subspace of polynomials.

(Alternatively, setting the coordinate vectors as rows of a matrix and finding a basis for its row space also works, and via the usual method – the nonzero rows of the row reduced echelon form form a basis for the row space – you would obtain $\{1 - t^2, t + 2t^2\}$ as a basis for the original subspace of polynomials.)
(b) Using coordinate vectors to go from \(P_2\) to \(\mathbb{R}^3\), this question is equivalent to finding the coordinates of the vector \([q(t)]_B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\) in the basis \(\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\} \). To do that, solve \(Ax = b\) for
\[
A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix},
\]
the solution \(x\) will give the weights that allow us write \(b\) as a linear combination of the columns of \(A\), which are exactly the coordinates that we are looking for. Solve the augmented system
\[
\begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 5 \\ 1 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}
\]
to get \(x = \begin{bmatrix} 6 \\ -1 \end{bmatrix}\), the coordinates are \((6, -1)\):
\[
q(t) = 6p_1(t) - p_2(t).
\]
(If you used the basis \(\{1 - t^2, t + 2t^2\}\), you would solve \(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} x = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix},\)
which with fewer computations yields \(x = \begin{bmatrix} 4 \\ 5 \end{bmatrix}\). In that case the coordinates of \(q(t) = 4(1 - t^2) + 5(t + 2t^2)\) in that basis would be \((4, 5)\).

3 (15 points). (a) Find the determinant of the matrix \(A = \begin{bmatrix} 100 & 200 & 300 \\ 101 & 201 & 301 \\ 102 & 202 & 345 \end{bmatrix}\).

(b) If \(B\) is a \(3 \times 3\) matrix and \(B^T = -B\), what is \(\det B\)? Justify your answer.

(c) If \(v_1, v_2, v_3\) are vectors in \(\mathbb{R}^3\) which are edges of a parallelepiped with volume 1, what are the possible values of \(\det \begin{bmatrix} 2v_1 & v_2 & 5v_3 \end{bmatrix}\)?

Solution

(a) Perform row operations that keep the determinant unchanged:
\[
\begin{vmatrix} 100 & 200 & 300 \\ 101 & 201 & 301 \\ 102 & 202 & 345 \end{vmatrix} = \begin{vmatrix} 100 & 200 & 300 \\ 1 & 1 & 1 \\ 2 & 2 & 45 \end{vmatrix} = \begin{vmatrix} 100 & 200 & 300 \\ 1 & 1 & 1 \\ 0 & 0 & 43 \end{vmatrix} = (-1)^{3+3} \cdot 43 \begin{vmatrix} 100 \\ 1 \\ 1 \end{vmatrix} = -4300.
\]
(b) If $B^T = -B$, then $\det(B^T) = \det(-B)$. Then we have

$$\det(B) = \det(B^T) = \det(-B) = (-1)^3 \det(B) = -\det(B),$$

where on the second to last equality we used that $B$ is $3 \times 3$. The only way to have $\det(B) = -\det(B)$ is if $\det(B) = 0$.

(c) $\det \begin{bmatrix} 2v_1 & v_2 & 5v_3 \end{bmatrix} = 2 \cdot 5 \det \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \pm 10$. The first step uses the fact that the determinant is linear on each column, and the second the fact that the volume of the parallelepiped determined by three vectors is equal to the absolute value (hence the $\pm$) of the determinant of the matrix that has those three vectors as columns.

4 (15 points). Which of the following subsets $W$ of the vector space $V$ of all $2 \times 2$ matrices are subspaces, and why or why not?

(a) The subset $W$ consisting of all matrices that have the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in their column space.

(b) The subset $W$ consisting of all matrices that have the vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ in their nullspace.

Solution

(a) The zero vector of $V$, which in this case is the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, is not contained in the subset $W$, because it does not have the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in its column space, so $W$ is not a subspace.

(b) For short, let us write $v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Note that the subset $W$ consists of the matrices $A$ such that $Av = 0$. It is a subspace because:

- The matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is contained in $W$, since $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} v = 0$.
- If $A$ and $B$ are contained in $W$, which means that $Av = 0$ and $Bv = 0$, then
  $$(A + B)v = Av + Bv = 0 + 0 = 0,$$
  so $(A + B)$ is also contained in $W$.
- If $c$ is a scalar and $A$ is contained in $W$, which means that $Av = 0$, then
  $$(cA)v = c(Av) = c0 = 0,$$
  so $cA$ is also contained in $W$.

5 (20 points). Let $V$ and $W$ be vector spaces and $T : V \to W$ be a linear transformation between them. Assume that $V$ is spanned by $\{v_1, \ldots, v_n\}$, and define $w_i = T(v_i)$ for $i = 1, \ldots, n$. 

(a) Show that $T(V)$ (also called Range($T$)) is spanned by $\{w_1, \ldots, w_n\}$.

(b) Why does this imply that $\dim(T(V)) \leq \dim V$?

**Solution**

(a) We must show that any element of $T(V)$, that is, any $T(x)$ with $x$ in $V$, can be written as a linear combination of $w_1, \ldots, w_n$. Because $\{v_1, \ldots, v_n\}$ spans $V$, we can write any $x$ in $V$ as a linear combination $x = c_1v_1 + \ldots + c_nv_n$. Then,

$$T(x) = T(c_1v_1 + \ldots + c_nv_n) = c_1T(v_1) + \ldots + c_nT(v_n) = c_1w_1 + \ldots + c_nw_n,$$

where the second step is by linearity of $T$.

(b) Let $k = \dim(V)$ and $\{b_1, \ldots, b_k\}$ be a basis for $V$. In particular this basis spans $V$, so by part (a) the set $\{T(b_1), \ldots, T(b_k)\}$ spans $T(V)$. As with all spanning sets, one can form a basis from it by dropping some vectors (possibly zero of them) until the remaining set is a basis for $T(V)$. The number of vectors in that remaining set will be $\dim(T(V))$ and will necessarily be smaller than or equal to the number of vectors that we started with, which was $k = \dim(V)$.

6 (10 points). Say if the following statements are true or false and give a justification.

(a) The rank of $A + B$ is less than or equal to the ranks of both $A$ and $B$.

(b) If $A$ is invertible, the row space of $A^{-1}$ must be the same as the row space of $A$.

**Solution**

(a) False. Take for example $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. We have rank $A = \text{rank } B = 1$ but rank $(A + B) = \text{rank } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2$.

(b) True. The row space of an invertible $n \times n$ matrix is $\mathbb{R}^n$, so Row($A$) = $\mathbb{R}^n$. But $A^{-1}$ is an $n \times n$ invertible matrix as well, so Row($A^{-1}$) = $\mathbb{R}^n$. 