1) (20 points) Let

\[ A = \begin{pmatrix} 1 & -2 & 1 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 7 & -5 & -1 \end{pmatrix}, \quad \vec{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{b}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}. \]

Find all solutions to the systems of equations a) \( A\vec{x} = 0 \), b) \( A\vec{x} = \vec{b}_1 \), c) \( A\vec{x} = \vec{b}_2 \), where \( \vec{x} \) is in \( \mathbb{R}^4 \). Also: d) find numbers \( c_1, c_2, c_3 \) such that the system \( A\vec{x} = \vec{b} \) has a solution if and only if \( b_1, b_2, b_3 \) satisfy the equation \( c_1b_1 + c_2b_2 + c_3b_3 = 0 \).

2) (15 points) Let

\[ A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \]

Either find \( A^{-1} \) or show that it doesn't exist.

3) (15 points) Let

\[ A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 4 & -2 \end{pmatrix} \]

a) Find a matrix \( B \) such that \( AB = I \), where \( I \) is the \( 2 \times 2 \) identity matrix.

b) Is the matrix \( A \) invertible?

OVER
4) (10 points) For each of the following statements say if it is true or false; give reasons if it is true, a counterexample if it is false.

a) If three vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) in \( \mathbb{R}^4 \) are linearly independent, then any two of them are also linearly independent.

b) If four vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \) span \( \mathbb{R}^3 \), then any three of them also span \( \mathbb{R}^3 \).

5) (10 points) Say, giving reasons, if the following statement is true always, sometimes (but not always), or never. If it is sometimes true, say for what vectors \( \vec{b} \) it is true, and why.

"If \( A \) is an \( m \times n \) matrix, and \( \vec{b} \) is in \( \mathbb{R}^m \) then the sum of two solutions of the system \( A\vec{x} = \vec{b} \), where \( \vec{x} \) is in \( \mathbb{R}^n \), is also a solution."

6) (20 points) Let \( R: \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear transformation which rotates each vector \( \vec{v} \) in \( \mathbb{R}^2 \) counterclockwise by the angle 45°. Let \( S: \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear transformation which reflects each vector \( \vec{v} \) in \( \mathbb{R}^2 \) in the \( x_1 \)-axis i.e. \( S \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = \left( \begin{array}{c} x_1 \\ -x_2 \end{array} \right) \). Let \( T: \mathbb{R}^2 \to \mathbb{R}^2 \) be the composition of \( R \) and \( S \) i.e. \( T(\vec{v}) = S(R(\vec{v})) \) for all \( \vec{v} \) in \( \mathbb{R}^2 \). Find the standard matrices \( A, B, C \) of the linear transformations \( R, S, T \). How can \( C \) be obtained from \( A \) and \( B \)? Verify this relationship directly for the three matrices you found.

Note: \( \cos 45° = 1/\sqrt{2} = \sin 45° \).

7) (10 points) Write down two \( 2 \times 2 \) matrices \( A \) and \( B \), neither of which is the 0-matrix, satisfying (simultaneously) the two conditions \( A + B = 0 \) and \( AB = 0 \).