MATHEMATICS 2210 - PRELIMINARY EXAM 2

SOLUTIONS

Exercise 1:

(1) We show that the set S of matrices that commute with N is stable under addition and scalar multiplication. If A and B are matrices in S, and k a scalar, then we show that $(kA+B)$ also commutes with N:

$$(kA+B)N = kAN + BN = kNA + NB = N(kA) + NB = N(kA+B),$$

the second equality is true because A and B are matrices in S, hence commute with N.

(2) Let A be a matrix in S. Assume A has matrix:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

The equality $AN = NA$ gives:

$$AN = \begin{bmatrix} b & c & a \\ e & f & d \\ h & i & g \end{bmatrix} = \begin{bmatrix} g & h & i \\ a & b & c \\ d & e & f \end{bmatrix} = NA,$$

so that $b = g = f$, $c = d = h$ and $a = e = i$. We conclude that a matrix A in S has the form:

$$A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}.$$

A basis of S is then given by:

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right).$$

(3) The space S has a basis with 3 elements, so has dimension 3. The space P_2 also has dimension 3 (with basis $(1, t, t^2)$ for instance). As two linear spaces with the same dimension are always isomorphic, S and P_2 are isomorphic.

Comments:

(a) S is a linear space, not a linear transformation! So e.g. "$Ker(S)$" makes no sense in (3).

(b) S is a subspace of $\mathbb{R}^{3\times 3}$, not of $\mathbb{R}^3$! So e.g. a basis for S consists of appropriate 3×3 matrices, not of vectors in $\mathbb{R}^3$.

(c) Note: two linear spaces of equal dimension are ALWAYS isomorphic to each other.

Exercise 2:

(1) The transformation T is linear. To prove it, consider two matrices A, A' in $\mathbb{R}^{2\times 2}$ and a scalar k. We have to show that $T(kA + A') = kT(A) + T(A')$. We denote $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$, and compute:

$$\begin{aligned} T(kA + A') &= T(\begin{bmatrix} ka + a' & kb + b' \\ kc + c' & kd + d' \end{bmatrix} \\ &= ((kc + c') - (kd - d'))t^4 + ((kc + c') + (kd - d'))t^2 \\ &\quad +(kc + c') + (kd - d') + 2(ka + a') \\ &= k((c - d)x^4 + (c + d)x^2 + c + d + 2a) \\ &\quad +((c' - d')x^4 + (c' + d')x^2 + c' + d' + 2a') \\ &= kT(A) + T(A') \end{aligned}$$

(2) We want to find V and W of dimension 3 so that $T : V \to W$ is an isomorphism. This forces W to be in the image of T. However, $Im(T)$ is included in $span(1, t^2, t^4)$ which has dimension 3. So we are forced to choose $W = span(1, t^2, t^4) = Im(T)$.

On the other hand, the kernel of T is:

$$Ker(T) = \{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}, b \in \mathbb{R}\} = span(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}).$$

Indeed, if $T(A) = 0$, then $c + d = c - d = 0$, so that $c = d = 0$, then as $c + d + 2a = 0$, we have $a = 0$ as well, only b can be chosen freely. So we choose V to be a complement of the kernel:

$$V = \{\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}, a, c, d \in \mathbb{R}\}.$$

As the intersection of V with $Ker(T)$ is just the zero matrix, the transformation $T : V \to W$ has kernel $\{0\}$, and as V and W have the same dimension, it is an isomorphism.

(3) The transformation S is not linear, because of the coefficient b^3 and d^2 for the polynomial $S(A)$. More precisely:

$$S(\begin{bmatrix} 0 & 0 \\ 0 & 1+2 \end{bmatrix}) = 9x \neq x + 4x = S(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}) + S(\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}).$$

Exercise 3:

(1) We compute:

$$\begin{aligned} T(1) &= 2 \times 1 - 1 - 1 = 0, \\ T(x) &= 2x - (x + 1) - (x - 1) = 0, \\ T(x^2) &= 2x^2 - (x + 1)^2 - (x - 1)^2 = 2, \\ T(x^3) &= 2x^3 - (x + 1)^3 - (x - 1)^3 = 6x. \end{aligned}$$

The matrix of T is the matrix with columns the coordinates of $T(1), T(x), T(x^2), T(x^3)$ in the basis $\mathcal{B} = (1, x, x^2, x^3)$, that is:

$$B = \begin{bmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(2) It is clear from the matrix B that:

$$Im(A) = span(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}),$$

which corresponds in the space P_3 to $Im(T) = span(1, t)$.

(3) Let Q be a polynomial in $Im(T) = span(1, t)$, say $Q(x) = \alpha + \beta x$. Using the matrix of T, we see that $T(a + bx + cx^2 + dx^3) = -2c - 6dx$. So that all polynomials $P(x) = a + bx + cx^2 + dx^3$ with $c = \frac{-\alpha}{2}$ and $d = \frac{-\beta}{6}$ verify $T(P(x)) = \alpha + \beta x$.

If the polynomial P also satisfies $P(0) = a = 0$ and $P'(0) = 2b = 0$, then the only possibility is $P(x) = \frac{-\alpha}{2}x^2 + \frac{-\beta}{6}x^3$.

(4) The matrix S of change of basis from $\mathcal{B}$ to $\mathcal{C}$ is:

$$S = \begin{bmatrix} [1]_{\mathcal{C}} & [x]_{\mathcal{C}} & [x^2]_{\mathcal{C}} & [x^3]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(Note that $1 = \frac{1}{2}(1 - x) + \frac{1}{2}(1 + x)$ and $x = \frac{-1}{2}(1 - x) + \frac{1}{2}(1 + x)$.) The matrix of T in the basis $\mathcal{C}$ is $C = SBS^{-1}$. We compute:

$$S^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, C = SBS^{-1} = \begin{bmatrix} 0 & 0 & -1 & 3 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Exercise 4:

(1) Call $\vec{v_1}, \vec{v_2}$ the columns of A. An orthonormal basis obtained by Gram-Schmidt process is:

$$\vec{u_1} = \frac{\vec{v_1}}{||\vec{v_1}||}, \vec{u_2} = \frac{\vec{v_2} - (\vec{v_2}.\vec{u_1})\vec{v_2}}{||\vec{v_2} - (\vec{v_2}.\vec{u_1})\vec{v_2}||}.$$

The computation gives:

$$\vec{u_1} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{u_2} = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ -1 \\ -1 \\ -1 \end{bmatrix}.$$

(2) If $Q = [\vec{u_1}, \vec{u_2}]$ denotes the matrix with columns $\vec{u_1}$ and $\vec{u_2}$. The matrix of the orthogonal projection onto $Im(A)$ is QQ^T. We compute:

$$QQ^T = \frac{1}{3}\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 2 \end{bmatrix}$$

(3) The closest possible vector $A\vec{x}$ in $Im(A)$ to $\vec{b}$ is the orthogonal projection of $\vec{b}$ onto $Im(A)$, that is

$$A\vec{x} = QQ^T\vec{b} = \frac{1}{3}\begin{bmatrix} -3 \\ 3 \\ 0 \\ -4 \end{bmatrix}.$$

This gives a new system which is consistent and can be solved as $x_1 = \frac{-1}{3}$, $x_2 = \frac{-1}{3}$. (Other solution: $\vec{x} = (A^TA)^{-1}A^T\vec{b}$.)

Comments:

Besides computational error, this exercise is rather well treated. Yet be careful with the size of the matrices: in (2), the projection is a transformation of $\mathbb{R}^4$, so the matrix should be 4×4 (not 2×2 as Q^TQ).

Exercise 5:

(a) A skew symmetric matrix satisfies $A = -A^T$, which can be rewritten $a_{ij} = -a_{ji}$ for all indices i, j. In particular, this forces the diagonal terms to be zero: ($a_{ii} = -a_{ii}$). A basis is given by E_{ij} for $i < j$ where E_{ij} is the matrix with all entries zero except the i, j entry which equals 1 and the j, i entry which equals -1.

(b) We consider the linear space of continuous functions on the interval $[0, 1]$, with the inner product:

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt.$$

The Cauchy-Schwartz formula $|\langle f, g\rangle| \leq ||f||.||g||$ gives:

$$|\int_0^1 t\sqrt{e^t}dt| \leq \sqrt{(\int_0^1 t^2dt)(\int_0^1 e^tdt)} = \sqrt{\frac{e-1}{3}},$$

for $f(t) = t$ and $g(t) = \sqrt{e^t}$.