1.9.7 \( T(e_1) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \) and \( T(e_2) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \). Therefore \( A = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \).

1.9.11 Find the standard matrix \( A \) of \( T \). We have \( T(e_1) = -e_1 \) and \( T(e_2) = -e_2 \), so \( A = -I \). But \( -I \) is a (clockwise or counterclockwise) rotation of \( \pi \) radians about the origin. Check this by plugging \( \pm \pi \) into the rotation matrix on pg. 73.

1.9.19 Use Example 5 as a template. \( T(x_1, x_2, x_3) = \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \).

1.9.23 (a) True, by Theorem 10 (the first paragraph in Sec. 1.9 also makes this clear).

(b) True. The standard matrix of a rotation about the origin is given on pg. 73.

(c) False. The composition of two linear transformations is a linear transformation (see Problem 1.9.36)

(d) False. This statement is satisfied by every linear transformation, but the onto property is stronger.

(e) False. Consider the \( 3 \times 2 \) matrix \( A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \). The columns of \( A \) are linearly independent, so \( x \mapsto Ax \) is one-to-one (Thm 12).

1.9.36 We check the linearity condition on pg. 67:

Let \( c, d \) be arbitrary scalars and let \( u, v \) be arbitrary vectors in \( \mathbb{R}^p \). Then

\[
T(S(cu + dv)) = T(cS(u) + dS(v)) = cT(S(u)) + dT(S(v))
\]

since \( S \) is linear.

Therefore \( x \mapsto T(S(x)) \) is a linear transformation.

2.1.1

\[-2A = \begin{bmatrix} -4 & 0 & 2 \\ -8 & 10 & -4 \end{bmatrix},\]

\[B - 2A = \begin{bmatrix} 3 & -5 & 3 \\ -7 & 6 & -7 \end{bmatrix},\]

\[AC \text{ is undefined}\]

\[CD = \begin{bmatrix} 1 & 13 \\ -7 & -6 \end{bmatrix}.
\]

2.1.15 (a) False. \( AB = \begin{bmatrix} Ab_1 & Ab_2 \end{bmatrix} \).

(b) False. Roles of \( A \) and \( B \) are flipped.

(c) True. Theorem 2b.

(d) True. Theorem 3b.

(e) False. The transpose of a product equals the product of transposes in reverse order.

2.1.17 By matrix multiplication, \( AB = \begin{bmatrix} Ab_1 & Ab_2 \end{bmatrix} \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix} \).

Solve \( Ab_1 = \begin{bmatrix} -1 \\ 6 \end{bmatrix} \) and \( Ab_2 = \begin{bmatrix} 2 \\ -9 \end{bmatrix} \) to get \( b_1 = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \) and \( b_2 = \begin{bmatrix} -8 \\ -5 \end{bmatrix} \).
2.1.25 Proof that \( C = D \): \( C = CI_m = C(AD) = CAD = (CA)D = I_nD = D \).

Proof that \( n = m \):
Let’s sketch the steps: first, use \( CA = I \) to show that \( A \) cannot have more columns than rows \((n \leq m)\). Then use \( AD = I \) to show that \( A \) cannot have more rows than columns \((m \leq n)\). These two results imply our desired conclusion \( m = n \).

Step 1: If \( Ax = 0 \), multiply on the left by \( C \) to get \( CAx = Ix = x = 0 \). Therefore \( A \) cannot have more columns than rows \((n \leq m)\).

Step 2: For any \( b \), \( A(Db) = (AD)b = Ib = b \). Therefore \( A \) has a pivot in every row, so \( A \) cannot have more rows than columns \((m \leq n)\).

2.1.33 Fix \( i, j \) and check the \( ij \)-th entry of the matrices \((AB)^T\) and \( B^TA^T\) separately:

The \( ij \)-th entry of \((AB)^T\) is the \( ji \)-th entry of \( AB \), which is the sum of products from the \( j \)-th row of \( A \) and the \( i \)-th column of \( B \): \( a_{j1}b_{1i} + \cdots + a_{jn}b_{ni} \).

The \( ij \)-th entry of \( B^TA^T \) is the sum of products from the \( i \)-th row of \( B^T \) (which is the \( i \)-th column of \( B \)) and the \( j \)-th column of \( A^T \) (which is the \( j \)-th row of \( A \)): \( b_{1i}a_{j1} + \cdots + b_{ni}a_{jn} \).

Comparing the terms in each sum, we see that the sums must be equal.

2.2.3 \( A^{-1} = \begin{bmatrix} 1 & 1 \\ \frac{-7}{5} & \frac{8}{5} \end{bmatrix} \).

2.2.4 \( A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{-7}{4} & \frac{3}{4} \end{bmatrix} \).

2.2.10 (a) False. The inverse of the product is the product of inverses in reverse order.
(b) True. Theorem 6a.
(c) True. Theorem 4.
(d) True. Theorem 7.
(e) False. The row operations reduce \( I_n \) to \( A^{-1} \), not the other way around.

2.2.11 Uniqueness: If \( X \) is a solution, multiply both sides of the equation by \( A^{-1} \) on the left to get \( X = A^{-1}B \).

Existence: To show that \( A^{-1}B \) is actually a solution, note that \( A(A^{-1}B) = (AA^{-1})B = IB = B \). So this matrix does satisfy the required equation.

2.2.19 Uniqueness: Assuming \( X \) satisfies the matrix equation, we have:

\[
C^{-1}(A + X)B^{-1} = I
\]

\[
A + X = CB
\]

\[
X = CB - A.
\]

Thus if the solution exists, it must be \( X = CB - A \).

Existence: Check that \( CB - A \) is actually a valid solution: \( C^{-1}(A + CB - A)B^{-1} = C^{-1}(CB)B^{-1} = I \).

2.2.32 \( A \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \). This matrix cannot be row-reduced to \( I_3 \), so it is not invertible.
2.2.37 One possibility by trial and error: \( C = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix} \). You can also do this rigorously by considering each element of \( C \) to be an unknown. Then you can set up 4 equations in 6 unknowns. Solve this system in the usual way and pick the free variables so that all six variables are 1, −1, 0.

2.3.1 Invertible. Determinant: \( ad - bc = -30 + 21 = -9 \neq 0 \). Or, notice that the columns are not multiples of each other (Theorem 8e).

2.3.2 Not invertible. Determinant: \( ad - bc = 36 - 36 = 0 \). Or, notice that the columns are multiples of each other (Theorem 8e).

2.3.4 Not invertible. Columns include the zero vector, so Theorem 8e fails. It is also clear that Theorem 8c fails (clearly this matrix will not have more than 2 pivots) as well as Theorem 8b.