MATH 2210 HOMEWORK 14 SOLUTIONS

Section 6.7

Problem 1: (a) \( ||x|| = \sqrt{\langle x, x \rangle} = \sqrt{4 \cdot 1 + 5 \cdot 1} = 3. \)
\( ||y|| = \sqrt{\langle y, y \rangle} = \sqrt{4 \cdot 5 + 5 \cdot (-1) \cdot (-1)} = \sqrt{105}. \)
\( |\langle x, y \rangle|^2 = |4 \cdot 1 \cdot 5 + 5 \cdot 1 \cdot (-1)|^2 = 15^2 = 225. \)

(b) A vector \((z_1, z_2)\) is orthogonal to \(y\) iff \(4z_1 + 5z_2 = 0\). That is, \(z_2 = -\frac{4}{5}z_1\). So the vectors orthogonal to \(y\) are exactly those contained in the span of \((5, -4)\).

Problem 4: \( \langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1) = -10 \)

Problem 14: To check that this is an inner product, we need to check that it is
(1) symmetric: \( \langle u, v \rangle = T(u) \cdot T(v) = T(v) \cdot T(u) = \langle v, u \rangle. \)
(2) respects addition: \( \langle u + v, w \rangle = T(u + v) \cdot T(w) = (T(u) + T(v)) \cdot T(w) = T(u) \cdot T(w) + T(v) \cdot T(w) = \langle u, w \rangle + \langle v, w \rangle. \)
(3) respects scalar multiplication: \( \langle cu, w \rangle = T(cu) \cdot T(w) = cT(u) \cdot T(w) = c\langle u, w \rangle. \)
(4) positive definite: \( \langle u, u \rangle = T(u) \cdot T(u) \geq 0. \) Also, \( \langle u, u \rangle = T(u) \cdot T(u) = 0 \) implies \( T(u) = 0, \) and so \( u = 0 \) because \( T \) is one-to-one.

Problem 26: We use the Gramm-Schmidt algorithm to form an orthogonal basis \(v_1, v_2, v_3: \) let \(v_1 = 1.\) Since \(\int_{-2}^{2} t dt = 0,\) we have \(v_2 = t.\) Finally,
\[ v_3 = t^2 - \frac{\langle t^2, t \rangle}{\langle t, t \rangle} t - \frac{\langle t^2, 1 \rangle}{\langle 1, 1 \rangle} \]

Section 7.1

Problem 12: To check if the matrix \(A\) is orthogonal, we check if \(A^T \cdot A = I.\) We compute: \(AA^T = A^T A = I,\) and so \(A\) is orthogonal with inverse \(A^T.\)

Problem 20: To find \(P,\) we compute that the eigenspace corresponding to the eigenvalue \(-3\) is the span of the vectors \((2, -1, 2)^T\) and \((-1, 2, 2)^T.\) Similarly, the eigenspace corresponding to the eigenvalue \(15\) is the span of the vector \((2, 2, -1)^T.\) Since the matrix is symmetric, we know these two subspaces are orthogonal. So to find an orthonormal eigenbasis of \(\mathbb{R}^3,\) it suffices to perform Gramm-Schmidt on each of these subspaces individually. We form the matrix \(P\) using these orthonormal eigenvectors, and the matrix \(D\) as the matrix with eigenvalues on the diagonal entries:
\[ P = \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}, \quad D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 15 \end{bmatrix}. \]
Problem 30: Let $A$ and $B$ be orthogonally diagonalizable; then $A$ and $B$ are symmetric. So $(AB)^T = (BA)^T = A^TB^T = AB$, so $AB$ is symmetric and thus orthogonally diagonalizable.

Problem 34: Let $u_1, u_2, u_3$ be as in example 3; then $A = 7u_1u_1^T + 7u_2u_2^T - 2u_3u_3^T$. 