6.1.11. The norm of the given vector \( \mathbf{v} \) is
\[
\| \mathbf{v} \| = \sqrt{\frac{49}{16} + \frac{1}{4} + 1} = \sqrt{\frac{69}{4}} = \frac{\sqrt{69}}{2}.
\]
The unit vector in this direction is
\[
\frac{\mathbf{v}}{\| \mathbf{v} \|} = 4 \frac{1}{\sqrt{69}} \begin{pmatrix} 7/4 \\ 1/2 \\ 1 \end{pmatrix} = \frac{\sqrt{69}}{69} \begin{pmatrix} 7 \\ 2 \\ 4 \end{pmatrix}.
\]

6.1.19(d). This statement is false. A counterexample is the \( 2 \times 2 \) matrix \( \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \). The column vector \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) is in the null space of \( A \), but it is not orthogonal to the second column of \( A \).

6.1.19(e). This statement is true. To show that \( \mathbf{x} \) is in \( W^\perp \), we need to show that it is orthogonal to every vector in \( W \). For every \( \mathbf{v} \) in \( W \), there exist scalars \( c_1, \ldots, c_p \) such that \( \mathbf{v} = c_1 \mathbf{v}_1 + \cdots + c_p \mathbf{v}_p \). Therefore
\[
\mathbf{x} \cdot \mathbf{v} = \mathbf{x} \cdot (c_1 \mathbf{v}_1 + \cdots + c_p \mathbf{v}_p) = c_1 (\mathbf{x} \cdot \mathbf{v}_1) + \cdots + c_p (\mathbf{x} \cdot \mathbf{v}_p) = 0,
\]
so \( \mathbf{x} \) is contained in \( W^\perp \).

6.1.30. We need to show that \( W^\perp \) contains the zero vector and that it is closed under addition and scalar multiplication:

- the zero vector is orthogonal to any vector in \( \mathbb{R}^n \), so it is orthogonal to any vector in \( W \).
- suppose that \( \mathbf{z} \) is a vector in \( W^\perp \) and \( c \) a scalar. Then for any \( \mathbf{w} \) in \( W \), we have
\[
(c \mathbf{z}) \cdot \mathbf{w} = c (\mathbf{z} \cdot \mathbf{w}) = 0,
\]
where we used the fact that \( \mathbf{z} \) is orthogonal to any vector in \( W \). We conclude that \( c \mathbf{z} \) is orthogonal to any vector of \( W \), so it is contained in \( W^\perp \).
- the case of addition is very similar. Suppose that \( \mathbf{z}_1 \) and \( \mathbf{z}_2 \) are two vectors in \( W^\perp \). Then for any \( \mathbf{w} \) in \( W \), we have
\[
(\mathbf{z}_1 + \mathbf{z}_2) \cdot \mathbf{w} = \mathbf{z}_1 \cdot \mathbf{w} + \mathbf{z}_2 \cdot \mathbf{w} = 0,
\]
where we used the fact that \( \mathbf{z}_1 \) and \( \mathbf{z}_2 \) are orthogonal to all vectors in \( W \). Since the same holds for their sum \( \mathbf{z}_1 + \mathbf{z}_2 \), we conclude that it is contained in \( W^\perp \).

This concludes our proof of the fact that \( W^\perp \) is a subspace of \( \mathbb{R}^n \).

6.1.31. If \( \mathbf{x} \) is in \( W^\perp \), it is orthogonal to any vector in \( W \). Since \( \mathbf{x} \) is also contained in \( W \), this implies that \( \mathbf{x} \) is orthogonal to itself. But then \( \mathbf{x} \cdot \mathbf{x} = 0 \) and therefore \( \mathbf{x} = 0 \) (see Theorem 1 on p. 333).
6.2.22. This set of vectors is orthonormal.

6.2.30. A square matrix is orthogonal if and only if it has orthonormal columns. Changing the order of the columns has no effect on that.

6.2.31. Suppose that \( L = \text{span}\{u\} = \text{span}\{u'\} \). Then there exists a nonzero scalar \( c \) such that \( u' = cu \). For any vector \( v \) in \( \mathbb{R}^n \), we have

\[
\frac{u' \cdot v}{u' \cdot u'} = \frac{(cu) \cdot v}{(cu) \cdot (cu)} = \frac{c^2 u \cdot v}{c^2 u \cdot u} = u \cdot u.
\]

6.2.33. For any \( y_1 \) and \( y_2 \) in \( \mathbb{R}^n \) and any scalars \( c_1 \) and \( c_2 \), we have

\[
\text{proj}_L(c_1y_1 + c_2y_2) = \frac{u \cdot (c_1y_1 + c_2y_2)}{u \cdot u}u = \frac{c_1(u \cdot y_1) + c_2(u \cdot y_2)}{u \cdot u}u = c_1 \text{proj}_L(y_1) + c_2 \text{proj}_L(y_2).
\]

6.2.34. We know from the previous problem that the projection onto \( L \) is linear. For any \( y_1 \) and \( y_2 \) in \( \mathbb{R}^n \) and any scalars \( c_1 \) and \( c_2 \), we have

\[
\text{refl}_L(c_1y_1 + c_2y_2) = 2 \text{proj}_L(c_1y_1 + c_2y_2) - (c_1y_1 + c_2y_2)
\]

\[
= 2 \left[ c_1 \text{proj}_L(y_1) + c_2 \text{proj}_L(y_2) \right] - (c_1y_1 + c_2y_2)
\]

\[
= c_1 \left[ 2 \text{proj}_L(y_1) - y_1 \right] + c_2 \left[ 2 \text{proj}_L(y_2) - y_2 \right]
\]

\[
= c_1 \text{refl}_L(y_1) + c_2 \text{refl}_L(y_2).
\]

6.3.12. The Best Approximation Theorem states that closest point to \( y \) in a subspace \( W \) is the orthogonal projection of \( y \) onto \( W \). The Orthogonal Decomposition Theorem gives us a way to compute this projection:

\[
\hat{y} = \frac{y \cdot v_1}{v_1 \cdot v_1}v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2}v_2
\]

We have

\[
y \cdot v_1 = 3 \cdot 1 + (-1) \cdot (-2) + 1 \cdot (-1) + 13 \cdot 2 = 30.
\]

\[
y \cdot v_2 = 3 \cdot (-4) + (-1) \cdot 1 + 0 \cdot 0 + 13 \cdot 3 = 26.
\]

\[
v_1 \cdot v_1 = 1^2 + (-2)^2 + (-1)^2 + 2^2 = 10
\]

\[
v_2 \cdot v_2 = (-4)^2 + 1^2 + 0^2 + 3^2 = 26
\]

Therefore we have

\[
\hat{y} = \frac{30}{10}v_1 + \frac{26}{26}v_2 = 3v_1 + v_2 = \begin{pmatrix} -1 \\ -5 \\ -3 \\ 9 \end{pmatrix}
\]
6.3.15. We start by computing the orthogonal projection of $y$ onto the plane spanned by $u_1$ and $u_2$ using the formula

$$\hat{y} = y \cdot u_1 u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{35}{35} u_1 + \frac{-28}{14} u_2 = u_1 - 2u_2 = \begin{pmatrix} 3 \\ -9 \\ -1 \end{pmatrix},$$

$$\hat{y} - y = \begin{pmatrix} 3 \\ -9 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ -9 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -6 \end{pmatrix}.$$  

The distance from $y$ to the plane is

$$||\hat{y} - y|| = \sqrt{4 + 36} = 2\sqrt{10}.$$  

6.3.17. It is easy to check that $\{u_1, u_2\}$ is an orthonormal basis for $W$. Therefore you should certainly get $U^T U = I_2$. We have


The matrix $UU^T$ represents the projection onto $W$ (see Theorem 10). Therefore

$$\proj_W y = UU^T y = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}.$$  

6.3.21.

(a) True: the proof is the same as in 6.1.19(e).

(b) True: this follows from the Orthogonal Decomposition Theorem. For every vector $y$ in $\mathbb{R}^n$, there is a unique decomposition $y = \hat{y} + z$ where $\hat{y} = \proj_W y$ is in $W$ and $z$ is in $W^\perp$. Therefore $z = y - \proj_W y$ is orthogonal to every vector in $W$.

(c) False: the Orthogonal Decomposition Theorem provides an easy answer. Any vector in $\mathbb{R}^n$ can be uniquely written as a sum of a vector in $W$ and a vector in $W^\perp$. The former is the called the projection onto $W$. It is defined without referring to any basis of $W$. The easiest way to compute it, however, is using a basis for $W$.

(d) True: use the Orthogonal Decomposition Theorem. There is only one way of writing $y$ as a sum of a vector in $W$ (the projection of $y$ onto $W$) and a vector in $W^\perp$. Clearly that is $y = y + 0$ because $y$ in $W$ and the zero vector $0$ is in $W^\perp$. Therefore $\proj_W y = y$.

(e) True: see Theorem 10.
6.4.16. Follow the steps outlined in the proof of Theorem 12. You get

\[ Q = \begin{pmatrix}
\frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \\
-\frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \\
0 & \frac{\sqrt{2}}{2} & 0 \\
\frac{1}{2} & \frac{\sqrt{2}}{4} & -\frac{1}{2} \\
\frac{1}{2} & \frac{\sqrt{2}}{4} & \frac{1}{2}
\end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix}
2 & 8 & 7 \\
0 & 2\sqrt{2} & 3\sqrt{2} \\
0 & 0 & 6
\end{pmatrix}.

6.4.18.

(a) False. One of the vectors \( v_i \) could be the zero vector. The statement is true if we add the condition that the vectors need to be nonzero.

(b) True. Suppose that \( x - \text{proj}_W x \) is zero. Then \( x = \text{proj}_W x \). Since the projection onto a subspace \( W \) is a vector in \( W \), this implies that \( x \) is in \( W \). Therefore, if \( x \) is not in \( W \), the vector \( x - \text{proj}_W x \) cannot be zero.

(c) True by the definition of QR-factorization.

6.4.22. We can prove this without using a basis. Let \( y_1 \) and \( y_2 \) be two vectors in \( \mathbb{R}^n \) and let \( c_1 \) and \( c_2 \) be scalars. The Orthogonal Decomposition Theorem states that there are unique vectors \( z_1 \) and \( z_2 \) in \( W^\perp \) such that \( y_1 = \text{proj}_W y_1 + z_1 \) and \( y_2 = \text{proj}_W y_2 + z_2 \). Then

\[ c_1 y_1 + c_2 y_2 = c_1 (\text{proj}_W y_1 + z_1) + c_2 (\text{proj}_W y_2 + z_2) \]
\[ = (c_1 \text{proj}_W y_1 + c_2 \text{proj}_W y_2) + (c_1 z_1 + c_2 z_2). \]

In the last expression, the first term is in \( W \) since both projections are in \( W \) and since \( W \) is a subspace of \( \mathbb{R}^n \). Similarly, the second term is in \( W^\perp \) since both \( z_1 \) and \( z_2 \) are in \( W^\perp \) and since \( W^\perp \) is a subspace of \( \mathbb{R}^n \). The Orthogonal Decomposition Theorem says that this decomposition of \( c_1 y_1 + c_2 y_2 \) is unique and that the term in \( W \) is exactly the projection of \( c_1 y_1 + c_2 y_2 \) onto \( W \). In other words, we have

\[ \text{proj}_W (c_1 y_1 + c_2 y_2) = c_1 \text{proj}_W y_1 + c_2 \text{proj}_W y_2. \]

This shows that the projection onto \( W \) is linear.