Section 2.3

14. $A^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ $\quad BC = \begin{bmatrix} 14 & 8 & 2 \end{bmatrix}$ $\quad BD = \begin{bmatrix} 6 \end{bmatrix}$

$C^2 = \begin{bmatrix} -2 & -2 & -2 \\ 4 & 1 & -2 \\ 10 & 4 & -2 \end{bmatrix}$ $\quad CD = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$ $\quad DE = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$

16. The answer is

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 3 & 4 & 7 & 9 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}.$$

20. If

$$\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix},$$

then

$$\begin{bmatrix} 2a+3c & 2b+3d \\ -3a+2c & -3b+2d \end{bmatrix} = \begin{bmatrix} 2a-3b & 3a+2b \\ 2c-3d & 3c+2d \end{bmatrix},$$

that is, $2a+3c = 2a-3b$, $2b+3d = 3a+2b$, $-3a+2c = 2c-3d$, $-3b+2d = 3c+2d$, which implies that $d = a$, and $c = -b$, and therefore, the matrices commuting with A are of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

52. $JH = HJ = \begin{bmatrix} 0.2 & -1.4 \\ 1.4 & 0.2 \end{bmatrix}$. $J = \sqrt{2} \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}$, $H = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, where $\theta = \cos^{-1}(0.8)$.

J is the counterclockwise rotation by $\pi/4$ combined with multiplication by $\sqrt{2}$, and H is the counterclockwise rotation by θ. Given a vector $\vec{v} \in \mathbb{R}^2$, to get $JH\vec{v}$, we first rotate $\vec{v}$ by angle θ to get $H\vec{v}$, then rotate $H\vec{v}$ by $\pi/4$, and finally multiply the vector by $\sqrt{2}$; to get $HJ\vec{v}$, we first rotate $\vec{v}$ by $\pi/4$ and multiply it by $\sqrt{2}$ to get $J\vec{v}$, then rotate the vector $J\vec{v}$ by angle θ.

60. Since $\operatorname{rank}\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\right) = 1 < 2$, it follows that $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is not invertible, so that there is no matrix X such that $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X = I_2$.

23 **6** $A^{-1}=\begin{bmatrix}1&-2&1\\0&1&-2\\0&0&1\end{bmatrix}$

16 $A^{-1}=\begin{bmatrix}-8&5\\5&-3\end{bmatrix} \Rightarrow \begin{cases}x_1=-8y_1+5y_2\\x_2=5y_1-3y_2\end{cases}$

30 $\begin{bmatrix}0&1&b\\-1&0&c\\-b&-c&0\end{bmatrix}\xleftrightarrow[R_2\cdot(-1)]{R_1\leftrightarrow R_2}\begin{bmatrix}1&0&-c\\0&1&b\\-b&-c&0\end{bmatrix}\xrightarrow{R_3+R_1\cdot b+R_2\cdot c}\begin{bmatrix}1&0&-c\\0&1&b\\0&0&0\end{bmatrix}\neq I_3$

$\Rightarrow$ The matrix is not invertible for any $b,c\in\mathbb{R}$

54 a) A – invertible if & only if $a,b,c\neq 0$, $A^{-1}=\begin{bmatrix}1/a&0&0\\0&1/b&0\\0&0&1/c\end{bmatrix}$

b) A – invertible if & only if all of its diagonal elements are nonzero.

40 WLOG assume that the first two columns of A are equal.
(Without loss of generality)

$$A=\begin{bmatrix}a_{11}&a_{11}&a_{13}&\cdots&a_{1n}\\a_{21}&a_{21}&a_{23}&\cdots&a_{2n}\\a_{31}&a_{31}&a_{33}&&\\\vdots&\vdots&\vdots&&\\a_{n1}&a_{n1}&a_{n3}&\cdots&a_{nn}\end{bmatrix}\xrightarrow[R_1/a_{11}]{}\ \xrightarrow{R_j-\frac{a_{j1}}{a_{11}}R_1}\begin{bmatrix}1&1&*&*&\cdots&*\\0&*&*&\cdots&*\\0&0&*&*&\cdots&*\\\vdots&\vdots&&&&\\0&0&*&*&\cdots&*\end{bmatrix}\rightarrow$$

$\Rightarrow \operatorname{ref}(A)\neq 0$ & $\operatorname{rank}(A)<n \Rightarrow A$ – not invertible.

q.e.d. □