Math 221 Fall 2007, Prelim I.
September 25, 2007
7:30pm-9:00pm

Solutions

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1. (15 points) A linear transformation T is defined as follows:

$$T \quad : \quad \mathbb{R}^3 \to \mathbb{R}^2$$
$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ x-z \end{bmatrix}$$

Find a basis of the kernel of T.

SOLUTION.

Let $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the vector in the kernel. We have that $T(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Hence:

$$\begin{bmatrix} x+2y \\ x-z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = -2y \text{ and } x = z \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \forall x \in \mathbb{R}$$

So the kernel is $Ker(T) = Span(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix})$, so a basis vector (in fact, the basis has only one vector) is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ $\square$.

2. (15 points) Check whether or not the set W is a subspace of $\mathbb{R}^3$. Justify your answer.

(a)

$$W = \left\{ \begin{bmatrix} 2x+y \\ y^2 \\ 0 \end{bmatrix} \mid x, y \in \mathbb{R} \right\} \subset \mathbb{R}^3$$

(b)

$$W = \left\{ \begin{bmatrix} x+y \\ 3y \\ 0 \end{bmatrix} \mid x, y \in \mathbb{R} \right\} \subset \mathbb{R}^3$$

SOLUTION.

(a) This is not a subspace, because, for example, if we choose the vector $w = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\cdot 0+1 \\ 1^2 \\ 0 \end{bmatrix} \in W$ and the scalar $c = -1$, then $c \cdot w = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \notin W$.

(b) This is a subspace because:

- $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+0 \\ 3\cdot 0 \\ 0 \end{bmatrix} \in W$

- Let $w_1 = \begin{bmatrix} x_1 + y_1 \\ 3y_1 \\ 0 \end{bmatrix}$ and $w_2 = \begin{bmatrix} x_2 + y_2 \\ 3y_2 \\ 0 \end{bmatrix}$ be two vectors in W and c_1, c_2 be two scalars. Then we have:

$$c_1 \cdot w_1 + c_2 \cdot w_2 = \begin{bmatrix} c_1x_1 + c_1y_1 \\ 3c_1y_1 \\ 0 \end{bmatrix} + \begin{bmatrix} c_2x_2 + c_2y_2 \\ 3c_2y_2 \\ 0 \end{bmatrix} = \begin{bmatrix} (c_1x_1 + c_2x_2) + (c_1y_1 + c + 2y_2) \\ 3(c_1y_1 + c_2y_2) \\ 0 \end{bmatrix} = \begin{bmatrix} x+y \\ 3y \\ 0 \end{bmatrix} \in W$$

where $x = c_1x_1 + c_2x_2$ and $y = c_1y_1 + c_2y_2 \in \mathbb{R}$.

Alternatively (shorter solution) one can notice that $\begin{bmatrix} x+y \\ 3y \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, so $W = Im(T)$, where $T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so it is a subspace, since the image of a linear map is a subspace (of the codomain, which is $\mathbb{R}^3$ in this case). $\square$

3. (15 points) Suppose the rank of a 100-by-200 matrix is 50. What is its nullity? Explain your answer.

   SOLUTION. The matrix represents a linear map $T : V \to W$, where $dim(V) = 200$ and $dim(W) = 100$. By the rank-nullity theorem we have that $dim(V) = dim(Im(T)) + dim(Ker(T))$. We know that the rank ($= dim(Im(T))$) is 50, so $dim(Ker(T)) = dim(V) - dim(Im(T)) = 200 - 50 = 150$. So the nullity ($= dim(Ker(T))$) is 150. $\square$

4. (15 points) Draw four vectors $\overrightarrow{v}_1, \overrightarrow{v}_2, \overrightarrow{v}_3, \overrightarrow{v}_4$ in $\mathbb{R}^2$ such that $\{\overrightarrow{v}_1, \overrightarrow{v}_2\}$ and $\{\overrightarrow{v}_3, \overrightarrow{v}_4\}$ are dependent, while all other pairs of vectors are independent.

   SOLUTION. Draw, for example, the vectors (written in coordinate form) $\vec{v}_1 = (1, 0)$, $\vec{v}_2 = (2, 0)$, $\vec{v}_3 = (0, 1)$ and $\vec{v}_4 = (0, 2)$. $\square$

5. (20 points) Let two linear transformations T_1 and T_2 are defined as follows:

$$T_1(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} -2x \\ x+y \\ y \end{bmatrix}, T_2(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x+z \\ z-2y \end{bmatrix}).$$

   (a) Find the matrix of the linear transformation $T_1 \circ T_2$.

   (b) Find the matrix of the linear transformation $T_2 \circ T_1$.

   (c) Which, if any, of $T_1 \circ T_2$ and $T_2 \circ T_1$ are invertible? Give reasons.

   SOLUTION.

   (a)

$$T_1 \circ T_2(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = T_1(\begin{bmatrix} x+z \\ z-2y \end{bmatrix}) = \begin{bmatrix} -2x - 2z \\ x - 2y + 2z \\ -2y + z \end{bmatrix}$$

   So the matrix will be $M_1 = \begin{bmatrix} -2 & 0 & -2 \\ 1 & -2 & 2 \\ 0 & -2 & 1 \end{bmatrix}$.

(b)

$$T_2 \circ T_1(\begin{bmatrix} x \\ y \end{bmatrix}) = T_2(\begin{bmatrix} -2x \\ x+y \\ y \end{bmatrix}) = \begin{bmatrix} -2x+y \\ -2x-y \end{bmatrix}$$

So the matrix will be $M_2 = \begin{bmatrix} -2 & 1 \\ -2 & -1 \end{bmatrix}$.

(c) M_1 is not invertible (check that its row reduced echelon form is not I_3 - it's a matrix that has only 0's on the last row), so $T_1 \circ T_2$ is not invertible.

M_2 is invertible (check that its row reduced echelon form is I_2), so the corresponding linear map $T_2 \circ T_1$ is invertible. □

6. Prove the following statements.

(a) (10 points) If A is an n by n matrix such that $A^2 = 0$, then $I_n - A$ is invertible, where I_n is the n by n identity matrix.

(b) (10 points) If A and B are invertible n by n matrices, then AB is similar to BA.

SOLUTION.

(a) We notice that $(I_n - A)(I_n + A) = (I_n + A)(I_n - A) = I_n^2 + A - A - A^2 = I_n^2 = I_n$, so the matrices $(I_n - A)$ and $(I_n + A)$ are inverses of one another. Hence $I_n - A$ is invertible. □

(b) We notice that we can write $AB = B^{-1} \cdot (BA) \cdot B$, hence AB is similar to BA. □