

## MATHEMATICS 2210 - PRELIMINARY EXAM 2

OCTOBER 30, 2008

*No books, notes or calculator. Show all work. Write clearly.*

*Good luck.*

The linear space of  $n \times m$  matrices is denoted by  $\mathbb{R}^{n \times m}$ . The linear space of polynomials of degree at most  $d$  is denoted by  $P_d$ .

### **Exercise 1:** (20 points)

Let  $S$  denote the set of  $3 \times 3$  matrices that commute with the matrix  $N$  where:

$$N = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

- (1) Show that  $S$  is a subspace of  $\mathbb{R}^{3 \times 3}$ .
- (2) Find a basis of  $S$ .
- (3) Is  $S$  isomorphic to the space  $P_2$  of polynomials of degree at most 2 ?

### **Exercise 2:** (20 points)

Consider the two following transformations  $T$  from  $\mathbb{R}^{2 \times 2}$  to  $P_4$  given by:

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (c - d)x^4 + (c + d)x^2 + c + d + 2a,$$

and  $S$  from  $\mathbb{R}^{2 \times 2}$  to  $P_3$  given by:

$$S\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (b + c)x^3 + b^3x^2 + d^2x + c - a.$$

- (1) Is  $T$  a linear transformation ? Justify.
- (2) Find linear subspaces  $V$  of  $\mathbb{R}^{2 \times 2}$  and  $W$  of  $P_4$ , both  $V$  and  $W$  of dimension 3, such that the transformation  $T : V \rightarrow W$  is an isomorphism.
- (3) Is  $S$  a linear transformation ? Justify.

*(continued on page 2)*

**Exercise 3:** (30 points)

Consider the linear transformation  $T : P_3 \rightarrow P_3$  which sends a polynomial  $P(x)$  to the polynomial:

$$T(P(x)) = Q(x) = 2P(x) - P(x+1) - P(x-1).$$

- (1) Find the matrix of  $T$  in the basis  $\mathcal{B} = (1, x, x^2, x^3)$  of  $P_3$ .
- (2) Find the image  $Im(T)$  of the transformation  $T$ .
- (3) Let  $Q$  be a polynomial in  $Im(T)$ , show there exists only one polynomial  $P$  in  $P_3$  such that  $T(P) = Q$  and  $P(0) = P'(0) = 0$ .
- (4) Find the matrix of  $T$  in the basis  $\mathcal{C} = (1-x, 1+x, x^2, x^3)$ .

**Exercise 4:** (30 points)

$$\text{Let } A = \begin{bmatrix} 2 & 1 \\ 0 & -3 \\ 2 & -2 \\ 2 & 4 \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ 2 \\ -1 \\ -2 \end{bmatrix}.$$

- (1) Find an orthonormal basis of  $Im(A)$ .
- (2) Find the matrix of the orthogonal projection onto  $Im(A)$ .
- (3) Find a vector  $\vec{x}$  in  $\mathbb{R}^2$  such that  $A\vec{x}$  is the closest possible to  $\vec{b}$  (such a vector is called a least squares solution of the system  $A\vec{x} = \vec{b}$ ).

**Exercise 5:** (bonus 10 points)

- (a) Give a basis of the space of  $n \times n$  symmetric matrices.
- (b) Prove that:

$$\int_0^1 t\sqrt{e^t} dt \leq \sqrt{\frac{e-1}{3}}.$$