Math 2210 - Prelim 1

September 23, 2008

1. Solve the following system of linear equations (20 points):

$$\begin{aligned} x_1 + 2x_2 - x_3 + 3x_4 &= 3 \\ 2x_1 + 4x_2 + 4x_3 + 3x_4 &= 9 \\ 3x_1 + 6x_2 - x_3 + 8x_4 &= 10 \end{aligned} \tag{1}$$

2. Are the following statements true or false (4 points each):

a. If T is a linear transformation and $T(v_1) = 0$ and $T(v_2) = 0$, then $T(2v_1 + v_2) = 0$.

b. If T is an invertible linear transformation, then the matrix representing it must be a square matrix.

c. If a system of linear equations has free variables, then it must have infinitely many solutions.

d. Two vectors in $\mathbb{R}^3$ always span a plane.

e. Consider a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$. If the image of T is a line, then the kernel of T is also a line.

3. Let

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 7 \\ 3 & 0 \\ 5 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & -2 \\ 5 & 3 \end{bmatrix}.$$

Decide which of the following products are possible and compute them (15 points):

$$A^2, \quad B^2, \quad AB \quad BA$$

4a. For what values of k are the following three vectors in $\mathbb{R}^3$ linearly dependent? Include all calculations leading to your conclusion. (10 points)

$$\begin{bmatrix} k \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 1 \\ -7 \end{bmatrix}$$

4b*. Consider a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by the matrix

$$\begin{bmatrix} 4 & n \\ n & 1 \\ 12 & 3n \end{bmatrix}.$$

For what values (if any) of n is the image of T a line? (* indicates a harder question) (10 points)

5. Consider a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by (25 points)

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + x_3 \\ x_1 + x_3 \\ x_1 + x_2 + 3x_3 \end{bmatrix}.$$

a. Show that T is linear and compute its matrix.

b. Determine the kernel and the image of T.

6 (BONUS - 10 points). Denote by V the collection of all $n \times n$ matrices. Suppose you have chosen one such matrix M. Consider a function $f : V \to V$ defined by $f(A) = AM + MA$. Show that f is a linear transformation.