HW 9

5.3 **4** Not orthogonal - column I is not perpendicular on column III.

8 Take $A = B = I_n \Rightarrow A + B = 2I_n = \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 2 \end{bmatrix}$ is not orthogonal.

34 $\left.\begin{array}{l} C_I \perp C_{III} \\ C_{II} \perp C_{III} \end{array}\right\} \Rightarrow c = d = 0. \Rightarrow A = \begin{bmatrix} a & b & 0 \\ 0 & 0 & 1 \\ d & e & 0 \end{bmatrix}$. Now $\begin{cases} a^2 + d^2 = 1 \\ b^2 + e^2 = 1 \\ ab + de = 0 \Rightarrow d = -\frac{ab}{e} \end{cases} \Rightarrow$

$\Rightarrow \frac{a^2 b^2}{e^2} + a^2 = 1 \Rightarrow \frac{a^2}{e^2} \underbrace{(b^2 + e^2)}_{=1} = 1 \Rightarrow a = \pm e \Rightarrow \begin{cases} a = e \\ b = -d \end{cases}$ or $\begin{cases} a = -e \\ b = d \end{cases}$

$\Rightarrow$ Can take $|a| = \cos\theta$, $|b| = \sin\theta$ $\Rightarrow A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 1 \\ \sin\theta & \cos\theta & 0 \end{bmatrix}$ or $A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \\ \sin\theta & -\cos\theta & 0 \end{bmatrix}$

40 Do Gram-Schmidt & get an orthonormal basis: $u_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$ & $u_2 = \begin{bmatrix} -1/10 \\ 7/10 \\ -7/10 \\ 1/10 \end{bmatrix}$.

$M = QQ^T$ $\quad Q = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$

$\Rightarrow M = \frac{1}{100} \begin{bmatrix} 26 & 18 & 32 & 24 \\ 18 & 74 & -24 & 32 \\ 32 & -24 & 74 & 18 \\ 24 & 32 & 18 & 26 \end{bmatrix}$

5.4 **16** A - $m \times n$ matrix $\Rightarrow$ $\left.\begin{array}{l} \dim(\operatorname{Im} A)^\perp = m - \dim(\operatorname{Im} A) = m - \operatorname{rank}(A) \\ \dim(\operatorname{Ker}(A^T)) = m - \operatorname{rank}(A^T) \end{array}\right\} \Rightarrow \operatorname{rank}(A) = \operatorname{rank}(A^T)$

20 $\vec{x}^* = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\vec{b} - A\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ (by solving $(A^T A)\vec{x} = A^T \vec{b}$)

5.5 **8** $\left.\begin{array}{l} T(u+v) = \langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle = T(u) + T(v) \\ T(ku) = \langle ku, w \rangle = k\langle u, w \rangle = kT(u) \end{array}\right\} \Rightarrow T$ - linear

If $w = 0 \Rightarrow \operatorname{Im}(T) = \{0\}$ & $\operatorname{Ker}(T) = V$.

If $w \neq 0 \Rightarrow \operatorname{Im}(T) = \mathbb{R}$ & $\operatorname{Ker}(T) = W^\perp$, where $W = \operatorname{Span}\{w\}$

10 If $g(t) = a + bt + ct^2$ is $\perp$ on $f(t) = t \Rightarrow \int_{-1}^{1} g(t) f(t) = 0 \Rightarrow \frac{2}{3} b = 0 \Rightarrow b = 0$.

$\Rightarrow \{1, t^2\}$ - basis for the fcts. $\perp$ on $f(t) = t$.

Apply Gram-Schmidt to $\{1, t^2\}$ & get: $g_1(t) = \frac{1}{\|1\|} = \frac{1}{\frac{1}{2}\int_{-1}^{1} 1\,dt} = 1$

& $g_2(t) = \frac{t^2 - \langle 1, t^2 \rangle \cdot 1}{\| t^2 - \langle 1, t^2 \rangle 1 \|} = \frac{\sqrt{5}}{2}(3t^2 - 1)$

14 a) Let $f(t)=(t-1)(t-2) \Rightarrow \langle f,f\rangle = f^2(1)+f^2(2)=0 \Rightarrow$ Not a dot product
But $f\neq 0$

b) Axioms a), b), c) are easy to prove.

If $\langle f,f\rangle = 0 \Rightarrow f^2(1)+f^2(2)+f^2(3)=0 \Rightarrow$ $\left.\begin{array}{l} 1,2,3 \text{ - roots of } f \\ f \text{ - has at most 2 distinct roots} \\ \deg f = 2 \end{array}\right\} \Rightarrow$

$\Rightarrow f=0 \Rightarrow$ Axiom d) - true

$\Rightarrow$ It is a dot product.

20 a) $\left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = x+2y=0 \Rightarrow x=-2y \Rightarrow$ This is a line spanned by $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ are already perpendicular $\Rightarrow$ Just need to normalize them.

$\left\|\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\| = \sqrt{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}} = 1 \Rightarrow \quad u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\left\|\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\| = \sqrt{\begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}} = \sqrt{4} = 2 \Rightarrow u_2 = \frac{1}{2}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.