HW 2

1.3 **14** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\cdot(-1)+2\cdot 2+3\cdot 1 \\ 2\cdot(-1)+3\cdot 2+4\cdot 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

24 By fact 1.3.4 $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow A\vec{x} = \vec{c}$ has a unique solution.

30

Take $\text{rref}(A) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. As $\begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix}$ is a sol. of $A\cdot\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$

$\Rightarrow \begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix}$ is a sol. of $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \vec{x} = \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix}$. $\Rightarrow \text{rref}(A)\begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} z \\ 0 \end{bmatrix} \Rightarrow$

$\Rightarrow z = 2$

We have $\left[A \,\middle|\, \begin{matrix} 2 \\ 0 \\ 1 \end{matrix}\right] \xrightarrow{\text{row operations}} \left[\begin{matrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \,\middle|\, \begin{matrix} 2 \\ 0 \\ 0 \end{matrix}\right]$. Doing the backward process $\Rightarrow R_3 + \frac{1}{2}R_1 \Rightarrow$ We get $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$.

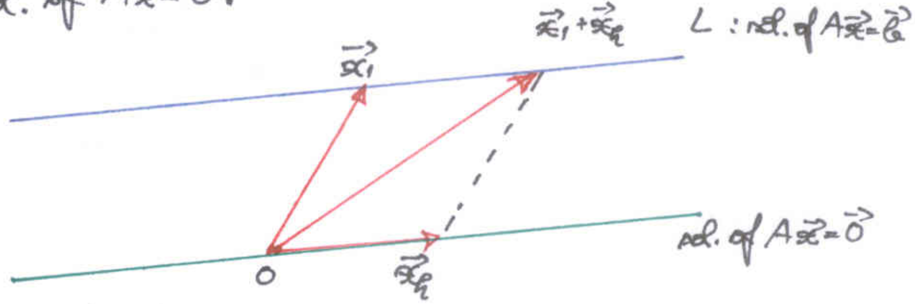
Note: The answer is not unique. Choosing other rref(A) will ~~get~~ result in a different result.

48 a) $A(\vec{x}_1 + \vec{x}_h) = A\vec{x}_1 + A\vec{x}_h = \vec{b} + \vec{0} = \vec{b}$

b) $A(\vec{x}_2 - \vec{x}_1) = A\vec{x}_1 - A\vec{x}_2 = \vec{b} - \vec{b} = \vec{0}$

c) From a) & b) $\Rightarrow$ The solutions of $A\vec{x} = \vec{b}$ are exactly the vectors of form $\vec{x}_1 + \vec{x}_h$, $\vec{x}_h$ - sol. of $A\vec{x} = \vec{0}$.

Geometrically:



58 $\begin{bmatrix} 3 \\ b \\ c \end{bmatrix} = \alpha_1 \cdot \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} + \alpha_3 \begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix}$ for ~~sol.~~ some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}$

$\Rightarrow$ The system $\left[\begin{matrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 2 & 4 & -2 \end{matrix} \,\middle|\, \begin{matrix} 3 \\ b \\ c \end{matrix}\right]$ has solution. We now reduce it & get:

$\begin{matrix} R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix} \Rightarrow \left[\begin{matrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \,\middle|\, \begin{matrix} 3 \\ b-9 \\ c-6 \end{matrix}\right]$ which has solution ~~iff~~ & only if $\boxed{\begin{matrix} b = 9 \\ c = 6 \end{matrix}}$

$\boxed{2.1}$

$\boxed{6}$ $x_1\begin{bmatrix}1\\2\\3\end{bmatrix}+x_2\begin{bmatrix}4\\5\\6\end{bmatrix}=\begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} \Rightarrow$ T - linear & its matrix is $\begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix}$

$\boxed{14}$ a) $\begin{bmatrix}2&3\\5&k\end{bmatrix}$ - invertible $\Leftrightarrow 2k-15\neq 0 \Leftrightarrow k\neq \frac{15}{2}$

b) $\begin{bmatrix}2&3\\5&k\end{bmatrix}^{-1}=\frac{1}{2k-15}\begin{bmatrix}k&-3\\-5&2\end{bmatrix}$ has integer entries.

CASE I: $k\in\mathbb{Z}\Rightarrow 2k-15\in\mathbb{Z}\Rightarrow (2k-15)$ divides $2,3,5$ primes $\Rightarrow$

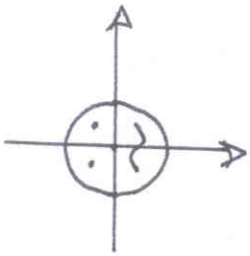
$\Rightarrow 2k-15=\pm 1\Rightarrow k=7$ or $k=8$

CASE II: $k\notin\mathbb{Z}\Rightarrow \frac{2}{2k-15}\in\mathbb{Z}\Leftrightarrow \frac{1}{2k-15}=m\in\mathbb{Z}\Rightarrow k=\frac{1}{2m}+\frac{15}{2}$

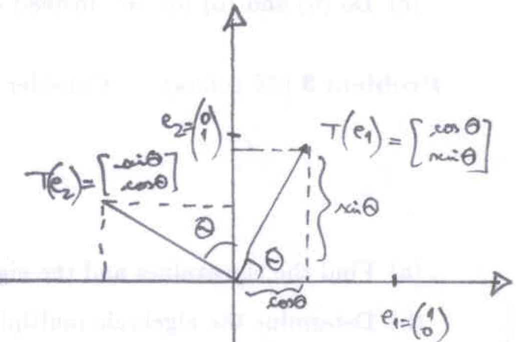
$\frac{k}{2k-15}=\frac{15m+1}{2}\in\mathbb{Z}\Rightarrow m$ - odd

So $k=\frac{1}{2m}+\frac{15}{2}$ with m - odd integer

$\boxed{24}$



$\boxed{34}$



$M=\left[T(e_1)\ T(e_2)\right]=\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}$

$\boxed{44}$ T - linear, because: $T(c_1\vec{x}_1+c_2\vec{x}_2)=\vec{v}\times(c_1\vec{x}_1+c_2\vec{x}_2)=$

$=\vec{v}\times(c_1\vec{x}_1)+\vec{v}\times(c_2\vec{x}_2)=c_1\cdot(\vec{v}\times\vec{x}_1)+c_2(\vec{v}\times\vec{x}_2)$

$=c_1T(\vec{x}_1)+c_2T(\vec{x}_2)$

The matrix is $\begin{bmatrix}T(e_1)&T(e_2)&T(e_3)\end{bmatrix}=\begin{bmatrix}0&-v_3&v_2\\v_3&0&-v_1\\-v_2&v_1&0\end{bmatrix}$

since $T(e_1)=v\times\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{bmatrix}0\\v_3\\-v_2\end{bmatrix}$, $T(e_2)=v\times\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{bmatrix}-v_3\\0\\v_1\end{bmatrix}$, $T(e_3)=v\times\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{bmatrix}v_2\\-v_1\\0\end{bmatrix}$