Math 221, Preliminary Exam 1

September 27, 2005

Name:

Lecture time:

Instructor:

Show your work and explain your solutions.

No calculators.

Problem 1. (10 points)

Solve the following system of linear equations using Gauss-Jordan elimination.

$$\begin{array}{rcrcrcr} -2x & - & 4y & + & 6z & = & -16 \\ x & + & 4y & - & 3z & = & 10 \\ 3x & + & 6y & - & 8z & = & 18 \end{array}$$

Solution: For clarity, we indicate every step of the Gaussian elimination:

$$\left[\begin{array}{ccc|c} -2 & -4 & +6 & -16 \\ 1 & 4 & -3 & 10 \\ 3 & 6 & -8 & 18 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 4 & -3 & 10 \\ -2 & -4 & +6 & -16 \\ 3 & 6 & -8 & 18 \end{array}\right] \begin{array}{l} \\ :2 \\ \\ \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 4 & -3 & 10 \\ -1 & -2 & 3 & -8 \\ 3 & 6 & -8 & 18 \end{array}\right] \begin{array}{l} \\ +I \\ -3\,I \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 4 & -3 & 10 \\ 0 & +2 & 0 & +2 \\ 0 & -6 & 1 & -12 \end{array}\right] \begin{array}{l} \\ \\ +3\,II \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 4 & -3 & 10 \\ 0 & +2 & 0 & +2 \\ 0 & 0 & 1 & -6 \end{array}\right] \begin{array}{l} \\ :2 \\ \\ \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 4 & -3 & 10 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -6 \end{array}\right] \begin{array}{l} +3\,III \\ \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 0 & -8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -6 \end{array}\right] \begin{array}{l} -4\,II \\ \\ \\ \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -12 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -6 \end{array}\right]$$

The system has a unique solution: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -12 \\ 1 \\ -6 \end{bmatrix}$.

Problem 2. (15 points)

(a) Find the matrix A of the following linear transformation.

$$\begin{aligned} y_1 &= x_1 + 2x_2 \\ y_2 &= x_1 + x_2 \\ y_3 &= 3x_3 \\ y_4 &= 2x_3 + 2x_4 \end{aligned}$$

(b) Find A^{-1}.

(c) Find A^2.

Solution: The matrix of the transformations is the block matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & C \end{array}\right].$$

To compute the inverse, we can use Gauss elimination:

$$[\,A \mid I\,] = \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & -2/3 & 1 \end{array}\right] \longrightarrow$$

$$\longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1/3 & 1/2 \end{array}\right] \Longrightarrow A^{-1} = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & -1/3 & 1/2 \end{bmatrix}.$$

Or we can use the fact that A is a block matrix, and apply the formula for the inverse for a 2×2 matrix: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
This gives:

$$A^{-1} = \left[\begin{array}{c|c} B^{-1} & 0 \\ \hline 0 & C^{-1} \end{array}\right] \quad \text{with} \quad B^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad C^{-1} = \begin{bmatrix} 1/3 & 0 \\ -1/3 & 1/2 \end{bmatrix}.$$

The second power of A can be computed in a similar way:

$$A^2 = \left[\begin{array}{c|c} B^2 & 0 \\ \hline 0 & C^2 \end{array}\right] \quad \text{with} \quad B^2 = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad C^{-1} = \begin{bmatrix} 9 & 0 \\ 10 & 4 \end{bmatrix}.$$

Problem 3. (20 points)

(a) Define the following two terms:

(i) rank of a matrix, and

(ii) kernel of a matrix.

Let $B = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 4 \end{bmatrix}$.

(b) Find the rank of the matrix B.

(c) Find the kernel of the matrix B.

Solution:
(a) Let A be an $m \times n$ matrix. The rank of A is the number of leading ones in the reduced row-echelon form of A. The Kernel of A is the set $\{\underline{x} \in \mathbb{R}^n \colon A\underline{x} = \underline{0}\}$.

(b) By Gauss elimination:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 4 \end{bmatrix} \begin{matrix} \\ -2I \end{matrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{matrix} \\ :2 \end{matrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} -II \\ \\ \end{matrix} \to \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The reduced row-echelon form of B has two leading ones, hence B has rank 2. The system $A\underline{x} = \underline{0}$ has infinitely many solutions, depending on 2 parameters.

(c) We find:

$$Ker(B) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3\alpha - 2\beta \\ \beta \\ \alpha \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} =$$

$$= Span \left(\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right).$$

Problem 4. (10 points)

Let $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$.

(a) Write v_3 as a linear combination of v_1 and v_2.

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be a linear transformation such that

$$T(v_1) = \begin{bmatrix} 0 \\ 2 \\ 3 \\ -2 \end{bmatrix} \text{ and } T(v_2) = \begin{bmatrix} 4 \\ -9 \\ 0 \\ -1 \end{bmatrix}.$$

Find $T(v_3)$.

Solution: (a) $v_3 = 2v_1 + 7v_2$. The coefficients of the linear combination can be found by inspection, or by solving the linear system with augmented matrix

$$\left[\begin{array}{cc|c} v_1 & v_2 & v_3 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 2 & 1 & 11 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 7 \end{array}\right].$$

(b) Because T is linear, we can write:

$$T(v_3) = T(2v_1 + 7v_2) = 2\,T(v_1) + 7\,T(v_2) = 2\begin{bmatrix} 0 \\ 2 \\ 3 \\ -2 \end{bmatrix} + 7\begin{bmatrix} 4 \\ -9 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 28 \\ -59 \\ 6 \\ -11 \end{bmatrix}.$$

Problem 5. (15 points)

Consider the following augmented matrix of a linear system.

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & a & b \end{array}\right]$$

For what a and b does this system have

(a) no solution,

(b) exactly one solution,

(c) infinitely many solutions?

Solution: If we apply Gauss elimination to this augmented matrix, we find:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & a & b \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & a-3 & b-2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a-5 & b-4 \end{array}\right].$$

We notice that:

- If $a = 5$ and $b \neq 4$, the last row becomes $\left[\begin{array}{ccc|c} 0 & 0 & 0 & \star \end{array}\right]$ with $\star \neq 0$, so the system has no solution.
- If $a = 5$ and $b = 4$, the last row becomes $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array}\right]$, so the system has infinitely many solutions.
- If $a \neq 5$, the next step of the Gauss elimination transforms the last row into $\left[\begin{array}{ccc|c} 0 & 0 & 1 & \frac{b-4}{a-5} \end{array}\right]$, so the system has exactly one solution (for every chosen value of b).

To summarize, we get:

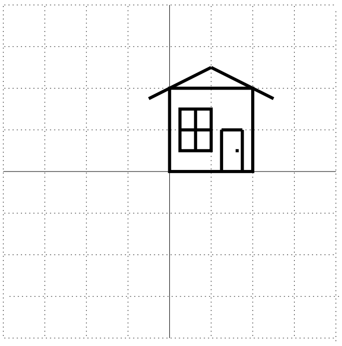
(a) no solution, for $a = 5$ and $b \neq 4$

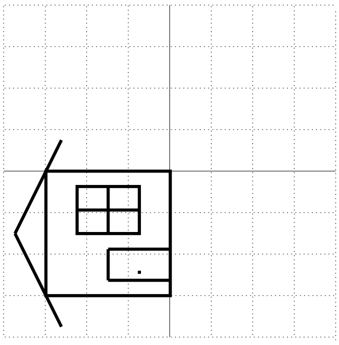
(b) infinitely many solutions, for $a = 5$ and $b = 4$

(c) exactly one solution, for $a \neq 5$.

Problem 6. (10 points)

Find the matrix of the linear transformation that sends the picture on the left to the picture on the right.





Solution: By inspection, we find: $T(e_1) = \begin{bmatrix} 0 \\ -3/2 \end{bmatrix}$, $T(e_2) = \begin{bmatrix} -3/2 \\ 0 \end{bmatrix}$.
Hence, the matrix of T is

$$\begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} = \begin{bmatrix} 0 & -3/2 \\ -3/2 & 0 \end{bmatrix} = 3/2 \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

We notice that the linear transformation T is the composition of the dilation by $3/2$ and the reflection about the line $y = -x$.

Problem 7. (20 points)

Determine whether the following statements are true or false, and justify your answer.

(a) The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$, defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 7x_1x_2 \\ -3x_2 \\ 4x_1 + x_2 \end{bmatrix},$$

is linear.

FALSE For instance, $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ -3 \\ 5 \end{bmatrix}$ and $T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 28 \\ -6 \\ 10 \end{bmatrix} \neq 2T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.

(b) There exist scalars a and b such that the matrix

$$\begin{bmatrix} 0 & 1 & a \\ -1 & 0 & b \\ -a & -b & 0 \end{bmatrix}$$

has rank 3. **FALSE** We can use Gauss elimination to prove that the matrix has *always* rank 2:

$$\begin{bmatrix} 0 & 1 & a \\ -1 & 0 & b \\ -a & -b & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & b \\ 0 & 1 & a \\ -a & -b & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -b \\ 0 & 1 & a \\ 0 & -b & -ab \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -b \\ 0 & 1 & a \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) There exists an invertible 5×5 matrix with two identical rows. **FALSE** If a 5×5 matrix A has two identical rows, the reduced row-echelon form of A contains the row $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, so the rank of A is less than 5. It follows that A cannot be invertible.

(d) If A is the matrix of a reflection, then $A^{-1} = A$. **TRUE** Because the inverse of a reflection is the reflection about the same line.

We can also prove this statement by a direct computation: the matrix of a reflection has the form $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, with $a^2 + b^2 = 1$, therefore: $A^{-1} = \frac{1}{-1}\begin{bmatrix} -a & -b \\ -b & a \end{bmatrix} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} = A$.

(e) If A is an invertible $n \times n$ matrix and $A^3 - 2A^2 = 0$, then $A = 2I_n$. **TRUE** Because if $A^3 = 2A^2 \Rightarrow A^3A^{-2} = 2A^2A^{-2} \Leftrightarrow A = 2I$.