Math 221 - Prelim 2- April 5, 2005

No notes. No calculators. No books.
WORK + ANSWER = CREDIT

1. (25) Suppose that V is a linear space with basis $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$. Let

$$\begin{aligned} \vec{w}_1 &= \vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4, \\ \vec{w}_2 &= \vec{v}_2 + \vec{v}_3, \\ \vec{w}_3 &= 2\vec{v}_1 - 2\vec{v}_3, \\ \vec{w}_4 &= \vec{v}_3 + \vec{v}_4. \end{aligned}$$

Is $\{\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4\}$ a basis of V? EXPLAIN.

Solution: V is a linear space with a four element basis $\{v_1, v_2, v_3, v_4\}$. Therefore, it has dimension 4, and it is sufficient to show that the four vectors w_1, w_2, w_3, and w_4 are either linearly independent or span V in order to show that they form a basis. There are several ways to show this.

One way is to look at the change of coordinate matrix from $\{w_1, w_2, w_3, w_4\}$ to $\{v_1, v_2, v_3, v_4\}$. Its columns are the w_i expressed as vectors in terms of the v_i:

$$S = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & -2 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Since the v_i do form a basis, so do the w_i iff this matrix is invertible. We can find the reduced row echelon form of S:

$$rref(S) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and see that S is not invertible. Indeed, we find that this transformation has a kernel spanned by $\begin{bmatrix} 1 \\ -1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$, or that $w_1 - w_2 - \frac{1}{2}w_3 + w_4 = 0$. Thus the vectors are not linearly independent, and not a basis. (Also, its image is not all of $V \cong \mathbb{R}^4$, and thus these do not span, either.)

One can also obtain S by attempting to solve the equation $c_1w_1 + c_2w_2 + c_3w_3 + c_4w_4 = 0$, substituting our known values for the w_i, and then setting the coefficient of each v_i to zero (since they are linearly independent). This produces four equations which yield the matrix S. This problem can also be solved by inspection, or by showing that a matrix A whose **rows** are the coefficients of the v_i in the given equation has a reduced row echelon form with one row all zeros, and thus its rows are not linearly independent. (Note that A is the transpose of S, and therefore invertible precisely when S is as well.)

2. (25) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation corresponding to multiplication by the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}.$$

Compute the matrix of T with respect to the basis $\mathfrak{B} = \{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}\}$.

Solution:

(2) We will solve this problem 2 ways — first by computing $S^{-1}AS$ and second column-by-column.

Method 1 (computing $S^{-1}AS$):

The matrix S is the change of basis matrix from the basis $\mathcal{B}$ to the standard basis. The columns of S are the vectors of the basis $\mathcal{B}$.

$S = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$

Since S is a 2×2 matrix, it is easy to determine the inverse of S:

$S^{-1} = \frac{1}{4-3}\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$

Then, the matrix B, representing the transformation T in the basis $\mathcal{B}$, is $S^{-1}AS$.

$B = S^{-1}AS = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -14 & -20 \\ 12 & 17 \end{bmatrix}$

Method 2 (column by column):

First we multiply each of the vectors in $\mathcal{B}$ by A:

$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \end{bmatrix}$

Next, we find the coordinates for these vectors in basis $\mathcal{B}$ (that is, we compute $\begin{bmatrix} -2 \\ 6 \end{bmatrix}_{\mathcal{B}}$ and $\begin{bmatrix} -3 \\ 8 \end{bmatrix}_{\mathcal{B}}$).

To compute $\begin{bmatrix} -2 \\ 6 \end{bmatrix}_{\mathcal{B}}$, we need to find a and b such that

$a \begin{bmatrix} 1 \\ 3 \end{bmatrix} + b \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$

This is equivalent to solving the following system of linear equations:

$$\begin{aligned} a + b &= -2 \\ 3a + 4b &= 6 \end{aligned}$$

We can set up an augmented matrix for this system and row-reduce:

$\left[\begin{array}{cc|c} 1 & 1 & -2 \\ 3 & 4 & 6 \end{array}\right] \begin{matrix} \\ -3I \end{matrix} \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & -2 \\ 0 & 1 & 12 \end{array}\right] \begin{matrix} -II \\ \\ \end{matrix} \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -14 \\ 0 & 1 & 12 \end{array}\right]$

Similarly, to compute $\begin{bmatrix} -3 \\ 8 \end{bmatrix}_{\mathcal{B}}$, we solve the following system of equations:

$$\begin{aligned} a+b &= -3 \\ 3a+4b &= 8 \end{aligned}$$

We set up an augmented matrix and row-reduce (note that the row reduction steps are exactly the same as the row reduction steps above):

$$\left[\begin{array}{cc|c} 1 & 1 & -3 \\ 3 & 4 & 8 \end{array}\right] \begin{array}{c} \\ -3I \end{array} \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & -3 \\ 0 & 1 & 17 \end{array}\right] \begin{array}{c} -II \\ \\ \end{array} \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -20 \\ 0 & 1 & 17 \end{array}\right]$$

This allows us to compute the matrix B:

$$B = \left[\left(A\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)_{\mathcal{B}}, \left(A\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right)_{\mathcal{B}}\right] = \left[\begin{bmatrix} -2 \\ 6 \end{bmatrix}_{\mathcal{B}}, \begin{bmatrix} -3 \\ 8 \end{bmatrix}_{\mathcal{B}}\right] = \begin{bmatrix} -14 & -20 \\ 12 & 17 \end{bmatrix}$$

3. (20) Be sure to EXPLAIN your answer to each of these short answer questions.

   (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is not in the image of T. List all possible dimensions for the kernel of T.

   Solution: Since the image of T is not all of $\mathbb{R}^3$, the rank of T is $0, 1$, or 2, but not 3. Hence, by the rank-nullity theorem, the kernel of T has dimension $1, 2$ or 3.

   (b) Let $S : \mathbb{R}^6 \to \mathbb{R}^5$ be a linear transformation. Suppose that the vectors

$$\begin{bmatrix} 2 \\ 0 \\ 0 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \\ 2 \\ 0 \end{bmatrix}$$

   are in the image of S. What is the maximum possible dimension for the kernel of S?

   Solution: Looking at the first three coordinates of the three vectors which are given as being in the image of S shows that they are linearly independent. Thus the dimension of the image of S is at least 3. By the rank-nullity theorem the dimension of the kernel of S is at most $6 - 3 = 3$. Note that many people said that the maximum was $5 - 3 = 2$. The rank-nullity theorem says that the $\dim Im(S) + \dim ker(S)$ equals the dimension of the *domain* of S.

(c) Let B be an $n \times n$ matrix which is both orthogonal and upper triangular. What is B?

Solution: B must be a diagonal matrix which has ± 1 in each diagonal entry. Since B is upper triangular the first column of B is $\begin{bmatrix} b_{11} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. As B is an orthogonal matrix, so the first column has unit length. Hence $b_{11} = \pm 1$. Again, since B is upper triangular the second column of B is $\begin{bmatrix} b_{12} \\ b_{22} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. As B is an orthogonal matrix, the dot product of the first and second column must be zero, hence $b_{12} = 0$. In addition, the second column must have unit length, so $b_{22} = \pm 1$. In each succeeding column the fact that the column must be orthogonal to the earlier ones implies that the entries are of the column are zero except for the diagonal entry. That the column must have unit length means that this entry must be ± 1.

4. (10) Let U, V, and W be linear spaces. Show that if T is a linear transformation from V to W, and L is a linear transformation from W to U, then the composite transformation $L \circ T$ (which first applies T, and after that applies L) from V to U is a linear tranformation. Note that the spaces are not assumed to be finite dimensional; in particular, you cannot use the matrix of a linear transformation.

Solution: Let $\vec{x}, \vec{y} \in W$ and let k be a scalar. $L \circ T$ is a linear transformation if and only if $L \circ T$ preserves vector addition and scalar multiplication.

$$(L\circ T)(\vec{x}+\vec{y}) \overset{\text{def}}{=} L(T(\vec{x}+\vec{y})) \overset{T \text{ is linear}}{=} L(T(\vec{x})+T(\vec{y})).$$

$$\overset{L \text{ is linear}}{=} L(T(\vec{x})+L(T(\vec{y})) \overset{\text{def}}{=} L\circ T(\vec{x})+L\circ T(\vec{y}).$$

So $L\circ T$ preserves vector addition.

$$(L\circ T)(k\vec{x}) \overset{\text{def}}{=} L(T(k\vec{x})) \overset{T \text{ is linear}}{=} L(k(T(\vec{x}))$$
$$\overset{L \text{ is linear}}{=} k(L(T(\vec{x})) \overset{\text{def}}{=} k(L\circ T(\vec{x})).$$

So $L\circ T$ preserves scalar multiplication.

5. (25) Find an orthonormal basis for the image of A and an orthonormal basis for the orthogonal complement of the image of A, where

$$A = \begin{bmatrix} 7 & 7 & 0 \\ -7 & 0 & 7 \\ 1 & 4 & 3 \\ -1 & 3 & 4 \end{bmatrix}.$$

Note: You need not simplify expressions of the form $\sqrt{n}$ when n is not a perfect square.

Solution: Let $\vec{v}_1$,$\vec{v}_2$ and $\vec{v}_3$ denote the column vectors of the matrix A. We can obtain an orthonormal basis for the im(A) using the Gram-Schmidt process:

$$\vec{w}_1 = \frac{\vec{v}_1}{||\vec{v}_1||} = \frac{1}{10}\begin{bmatrix} 7 \\ -7 \\ 1 \\ -1 \end{bmatrix}.$$

Next we compute

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2\cdot\vec{w}_1)\vec{w}_1 = \vec{v}_2 - 5\vec{w}_1 = \frac{1}{2}\begin{bmatrix} 7 \\ 7 \\ 7 \\ 7 \end{bmatrix},$$

which gives us

$$\vec{w}_2 = \frac{\vec{v}_2^{\perp}}{||\vec{v}_2^{\perp}||} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Computing $\vec{v}_3^{\perp}$ yields

$$\vec{v}_3^{\perp} = \vec{v}_3 - (\vec{v}_3 \cdot \vec{w}_1)\vec{w}_1 - (\vec{v}_3 \cdot \vec{w}_2)\vec{w}_2 = \vec{v}_3 + 5\vec{w}_1 - 7\vec{w}_2 = \vec{0}.$$

This shows that the vector v_3 is redundant and that $\{\vec{w}_1, \vec{w}_2\}$ is an orthonormal basis for im$A$.

The orthogonal complement of the image of A is the same as the kernel of A^T

$$(\text{im}A)^{\perp} = \ker A^T = \ker \begin{bmatrix} 7 & -7 & 1 & -1 \\ 7 & 0 & 4 & 3 \\ 0 & 7 & 3 & 4 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & 4/7 & 3/7 \\ 0 & 1 & 3/7 & 4/7 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus the kernel of A^T is spanned by $\begin{bmatrix} -4/7 \\ -3/7 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3/7 \\ -4/7 \\ 0 \\ 1 \end{bmatrix}$. Running Gram-Schmidt process on these 2 vectors will give us an ortonormal basis for $\ker A^T$, which involve many square roots. If we start with another basis of the kernel namely

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3/7 \\ -4/7 \\ 0 \\ 1 \end{bmatrix} \right\}$$

then the Gram-Schmidt process will yield much nicer orthonormal basis for $\ker A^T = (\text{im}A)^{\perp}$

$$w_3 = \frac{1}{2}\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad w_4 = \frac{1}{10}\begin{bmatrix} -1 \\ 1 \\ 7 \\ -7 \end{bmatrix}.$$

6. (25) Denote by P_2 the space of polynomials of degree ≤ 2. Find the matrix of of the linear transformation

$$T(f(t)) = f(2t-1)$$

from P_2 to P_2, with respect to the basis $\mathcal{B} = (1, t-1, (t-1)^2)$. Is T an isomorphism?

We have

$$\begin{aligned} T(1) &= 1 \\ T(t-1) &= (2t-1) - 1 = 2(t-1) \\ T\left((t-1)^2\right) &= \left((2t-1)-1\right)^2 = \left(2(t-1)\right)^2 = 4(t-1)^2. \end{aligned}$$

This shows that 1, $t-1$ and $(t-1)^2$ are eigenvectors for T and that the matrix of T in the basis $\mathcal{B}$ is diagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix},$$

since the diagonal entries are non zero this matrix is invertible which shows that T is an isomorphism.

7. (20) Let $\vec{x}, \vec{y}, \vec{z}$ be three vectors in $\mathbb{R}^2$. Assume that

$$||\vec{x}|| = 1, ||\vec{y}|| = 2, ||\vec{z}|| = 3, \vec{x} \cdot \vec{y} = 1 \text{ and } \vec{y} \cdot \vec{z} = 3\sqrt{3}.$$

(a) What is the angle (in radians) between $\vec{x}$ and $\vec{y}$?

(b) What are the possible values of $\vec{x} \cdot \vec{z}$?

Solution: a) The angle θ between the vectors $\vec{x}$ and $\vec{y}$ can be determined using the dot product

$$\cos\theta = \frac{\vec{x} \cdot \vec{y}}{||\vec{x}||.||\vec{y}||} = \frac{1}{2}.$$

Therefore $\theta = \cos^{-1} 1/2 = \pi/3 = 60^o$.

b) a) The angle ϕ between the vectors $\vec{y}$ and $\vec{z}$ is

$$\cos\phi = \frac{\vec{y} \cdot \vec{z}}{||\vec{y}||.||\vec{z}||} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2},$$

i.e., $\phi = \cos^{-1}\sqrt{3}/2 = \pi/6 = 30^o$.

The vectors $\vec{x}$, $\vec{y}$ and $\vec{z}$ lie in a plane, thus the angle psi between $\vec{x}$ and $\vec{z}$ is either $\theta + \phi = \pi/2$ or $\theta - \phi = \pi/6$. These two possibilities give us two possible values for the dot product $\vec{x} \cdot \vec{z}$:

$$\vec{x} \cdot \vec{z} = ||\vec{x}||.||\vec{z}|| \cos \pi/2 = 0 \quad \text{or} \quad \vec{x} \cdot \vec{z} = ||\vec{x}||.||\vec{z}|| \cos \pi/6 = \frac{3\sqrt{3}}{2}.$$