Math 221 - Linear algebra

Final Exam Dec. 7, 2006

No notes. No calculators. No books.
WORK + ANSWER + GOOD PRESENTATION = CREDIT

Problem 1. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation given by the matrix (with respect to the standard basis)

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Give a basis of the kernel of T and a basis of the image of T.

Problem 2. Let P_d be the vector space of the polynomials in the variable t with real coefficients of degree less or equal to d. Let T bet the linear transformation form P_2 to P_2 defined by

$$T(f)(t) = f(t) + (t+1)f'(t-1) + (t^2-1)f''(t)$$

a) Find the matrix of T in the basis $\mathfrak{B} = \{1, t, t^2\}$ and use it to determine whether T is an isomorphism.

b) Compute the matrix of T in the basis $\mathfrak{B} = \{1, t+1, (t+1)^2\}$ using two different methods.

Problem 3. Let P_d be the vector space of the polynomials in the variable t with real coefficients of degree less or equal to d. On P_d we consider the scalar product

$$(f, g) = \int_0^1 f(t)g(t)dt.$$

Let $\pi : P_2 \to P_1$ be the orthogonal projection from P_2 onto P_1. What is the dimension of the kernel of π? Compute $\pi(t^2 - t + 1/6)$.

Problem 4. Compute the determinant and the adjoint of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 5 & 2 & 1 & 7 \\ 2 & 0 & 4 & 5 \\ 3 & 1 & 1 & 7 \end{bmatrix}$$

and use them find A^{-1}.

Problem 5. a) Find (if possible) a diagonal matrix similar to the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 1 & 1 & -1 \\ -1 & -2 & 4 & -1 \\ 2 & -2 & 2 & 0 \end{bmatrix}$$

b) Is it possible to diagonalize the matrix

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 3 & 0 \end{bmatrix}$$

over the real numbers? over the complex numbers?

Problem 6. Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix (with respect to the standard basis)

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Let $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ be the solution of the differential equation

$$\frac{d}{dt}x(t) = Ax(t)$$

with initial condition

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Compute $x_1(t)$ and $x_2(t)$.