Math 221 Prelim 2

March 29, 2007

Name: ____________________

Instructor: ____________________

Section: ____________________

INSTRUCTIONS — READ THIS NOW

- This test has 5 problems on 6 pages (counting this one) worth a total of 50 points.
- Write your name, your instructor's name, and your section number **right now**.
- Show your work/explanation. To receive full credit, your answers must be neatly written and logically organized. If you need more space, write on the back side of the preceding sheet, but be sure to clearly label your work.
- Notes, books, "cheat sheets", cell phones, and personal audio players are not allowed. Calculators are neither needed nor permitted.
- This is a 90 minute test.

OFFICIAL USE ONLY

1. ______
2. ______
3. ______
4. ______
5. ______

Total: ______

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1. [**10pts (2pts each)**] Decide whether each of the following statements is TRUE or FALSE. Please write only TRUE or FALSE — you are not required to explain your answer.

   (a) The matrix
   $$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}.$$
   is orthogonal.
   FALSE. The columns are not unit vectors.

   (b) The transformation $T(f(t)) = f(173)$ from P_7 to $\mathbb{R}$ is linear.
   TRUE.

   (c) If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^8$, then $\|\vec{u}\|\|\vec{v}\| \geq \vec{u} \cdot \vec{v}$
   TRUE. Follows from 5.1.11.

   (d) If V is a linear space (vector space) of dimension 7, then it is isomorphic to $\mathbb{R}^7$.
   TRUE. See 4.2.3.

   (e) If A, B, and C are 3×3 matrices, such that C is invertible and A is similar to B, then CAC^{-1} is similar to B.
   TRUE. CAC^{-1} is similar to A, and A is similar to B, hence CAC^{-1} is similar to B. (See 3.4.6.)

2. (a) **[2pts]** Suppose A is an $n \times m$ matrix. The Rank-Nullity Theorem gives what relationship between (some of) n, m, $\dim(\mathrm{im}(A))$, and $\dim(\ker(A))$?

$$\dim(\ker(A)) + \dim(\mathrm{im}(A)) \;=\; m.$$

(b) **[5pts]** Explain why the first two columns of

$$A \;=\; \begin{bmatrix} 3 & 1 & 1 & -5 \\ -1 & 0 & -1 & 2 \\ 2 & 4 & -6 & 0 \end{bmatrix}.$$

form a basis for its image. Determine $\dim(\ker(A))$.

We have

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first two columns of $\mathrm{rref}(A)$ contain the leading 1's. Therefore, the first two columns of A form a basis of $\mathrm{im}(A)$, and $\dim(\mathrm{im}(A)) = 2$. So by the Rank-Nullity Theorem,

$$\dim(\ker(A)) = (\#\text{ columns of } A) - \dim(\mathrm{im}(A)) \;=\; 4 - 2 \;=\; 2.$$

(c) **[3pts]** Find a basis for the orthogonal complement of the image of the matrix A of part (b).

A vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in $(\mathrm{im}(A))^\perp$ if and only if $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = 0$; that is, $3a - b + 2c = 0$ and $a + 4c = 0$. As

$$\mathrm{rref}\left(\begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 10 \end{bmatrix},$$

the solutions of this pair of equations are $a = -4k$, $b = -10k$, $c = k$ for any $k \in \mathbb{R}$, and so $\begin{bmatrix} -4 \\ -10 \\ 1 \end{bmatrix}$ is a basis for $(\mathrm{im}(A))^\perp$.

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3. Let $L: P_2 \longrightarrow P_2$ be the linear transformation

$$L(f) = \left(\int_0^1 f(t)dt\right) x^2 + \frac{d^2}{dx^2} f(x),$$

where P_2 is the linear space (vector space) of all polynomials $f(x)$ of degree at most 2 with real coefficients.

(a) **[3pts]** Calculate $L(ax^2 + bx + c)$ where a, b, c are real numbers.

$$L(ax^2+bx+c) = \left(\int_0^1 (at^2 + bt + c)dt\right) x^2 + \frac{d^2}{dx^2}(ax^2+bx+c) = \left(\frac{a}{3} + \frac{b}{2} + c\right) x^2 + 2a.$$

(b) **[4pts]** Find a basis for $\ker(L)$. What is the dimension of $\ker(L)$?

By the calculation above, $L(ax^2 + bx + c) = 0$ if and only if $a = 0$ and $c = -b/2$. So,

$$\ker(L) = \{ \, bx - (b/2) \mid b \in \mathbb{R} \, \}$$

and the polynomial $x - (1/2)$ (on its own) forms a basis for $\ker(L)$. It follows that $\dim(\ker(L)) = 1$.

(c) **[3pts]** For $\mathcal{B} = (x^2, x, 1)$, give the $\mathcal{B}$-matrix for L. (That is, the matrix that transforms $[f]_\mathcal{B}$ into $[L(f)]_\mathcal{B}$, for all $f \in P_2$.)

The $\mathcal{B}$-matrix for L is

$$\begin{bmatrix} [L(x^2)]_\mathcal{B} & [L(x)]_\mathcal{B} & [L(1)]_\mathcal{B} \end{bmatrix} = \begin{bmatrix} 1/3 & 1/2 & 1 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

4. (a) **[4pts]** What does it mean to say that vectors $\overrightarrow{v_1}$, $\overrightarrow{v_2}$, ..., $\overrightarrow{v_m}$ in $\mathbb{R}^n$ are *orthonormal*?

Vectors $\overrightarrow{v_1}$, $\overrightarrow{v_2}$, ..., $\overrightarrow{v_m}$ in $\mathbb{R}^n$ are *orthonormal* when they are all unit vectors (that is, $\overrightarrow{v_i} \cdot \overrightarrow{v_i} = 1$ for all i) and are mutually orthogonal (that is, $\overrightarrow{v_i} \cdot \overrightarrow{v_j} = 0$ for all $i \neq j$).

(b) **[6pts]** Show that if $\overrightarrow{v_1}$, $\overrightarrow{v_2}$, ..., $\overrightarrow{v_m}$ in $\mathbb{R}^n$ are orthonormal, then they are linearly independent.

We recall that $\overrightarrow{v_1}$, $\overrightarrow{v_2}$, ..., $\overrightarrow{v_m}$ are linearly independent when the only expression

$$c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + \ldots + c_m\overrightarrow{v_m} = \overrightarrow{0}$$

for $\overrightarrow{0}$ as a linear combination of $\overrightarrow{v_1}$, $\overrightarrow{v_2}$, ..., $\overrightarrow{v_m}$, has $c_1 = c_2 = \cdots = c_m = 0$.

So suppose $c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + \ldots + c_m\overrightarrow{v_m} = \overrightarrow{0}$ for some real numbers $c_1, c_2, \ldots, c_m$. Then for $i = 1, 2, \ldots, m$,

$$\begin{aligned} 0 = v_i \cdot \overrightarrow{0} &= \overrightarrow{v_i} \cdot (c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + \ldots + c_m\overrightarrow{v_m}) \\ &= c_1(\overrightarrow{v_i} \cdot \overrightarrow{v_1}) + c_2(\overrightarrow{v_i} \cdot \overrightarrow{v_2}) + \ldots + c_m(\overrightarrow{v_i} \cdot \overrightarrow{v_m}) \\ &\overset{(*)}{=} c_i(\overrightarrow{v_i} \cdot \overrightarrow{v_i}) \\ &\overset{(**)}{=} c_i. \end{aligned}$$

(Equality (*) is a consequence of orthonormal vectors being mutually orthogonal, and equality (**) is because othonormal vectors are unit vectors.) So $c_1 = c_2 = \cdots = c_m = 0$ and $\overrightarrow{v_1}$, $\overrightarrow{v_2}$, ..., $\overrightarrow{v_m}$ are linearly independent.

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5. **[10pts]** Find an orthonormal basis of the subspace V of $\mathbb{R}^4$ with basis

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

Applying the Gram-Schmidt process, we obtain the orthonormal basis

$$\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -\frac{1}{\sqrt{15}} \\ \frac{2}{\sqrt{15}} \\ -\frac{1}{\sqrt{15}} \\ \frac{3}{\sqrt{15}} \end{bmatrix}, \quad \begin{bmatrix} -\frac{4}{\sqrt{35}} \\ \frac{3}{\sqrt{35}} \\ \frac{1}{\sqrt{35}} \\ -\frac{3}{\sqrt{35}} \end{bmatrix}.$$

STOP. THIS IS THE LAST PAGE.