Math 221 - Prelim 2- April 5, 2005

No notes. No calculators. No books.
WORK + ANSWER = CREDIT

1. (25) Suppose that V is a linear space with basis $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$. Let

$$\begin{aligned} \vec{w}_1 &= \vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4, \\ \vec{w}_2 &= \vec{v}_2 + \vec{v}_3, \\ \vec{w}_3 &= 2\vec{v}_1 - 2\vec{v}_3, \\ \vec{w}_4 &= \vec{v}_3 + \vec{v}_4. \end{aligned}$$

   Is $\{\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4\}$ a basis of V? EXPLAIN.

2. (25) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation corresponding to multiplication by the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}.$$

   Compute the matrix of T with respect to the basis $\mathfrak{B} = \{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}\}$.

3. (20) Be sure to EXPLAIN your answer to each of these short answer questions.

   (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is not in the image of T. List all possible dimensions for the kernel of T.

   (b) Let $S : \mathbb{R}^6 \to \mathbb{R}^5$ be a linear transformation. Suppose that the vectors

$$\begin{bmatrix} 2 \\ 0 \\ 0 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \\ 2 \\ 0 \end{bmatrix}$$

   are in the image of S. What is the maximum possible dimension for the kernel of S?

(c) Let B be an $n \times n$ matrix which is both orthogonal and upper triangular. What is B?

4. (10) Let U, V, and W be linear spaces. Show that if T is a linear transformation from V to W, and L is a linear transformation from W to U, then the composite transformation $L \circ T$ (which first applies T, and after that applies L) from V to U is a linear tranformation. Note that the spaces are not assumed to be finite dimensional; in particular, you cannot use the matrix of a linear transformation.

5. (25) Find an orthonormal basis for the image of A and an orthonormal basis for the orthogonal complement of the image of A, where

$$A = \begin{bmatrix} 7 & 7 & 0 \\ -7 & 0 & 7 \\ 1 & 4 & 3 \\ -1 & 3 & 4 \end{bmatrix}.$$

Note: You need not simplify expressions of the form $\sqrt{n}$ when n is not a perfect square.

6. (25) Denote by P_2 the space of polynomials of degree ≤ 2. Find the matrix of of the linear transformation

$$T(f(t)) = f(2t - 1)$$

from P_2 to P_2, with respect to the basis $\mathcal{B} = (1, t - 1, (t - 1)^2)$. Is T an isomorphism?

7. (20) Let $\vec{x}, \vec{y}, \vec{z}$ be three vectors in $\mathbb{R}^2$. Assume that

$$||\vec{x}|| = 1, ||\vec{y}|| = 2, ||\vec{z}|| = 3, \vec{x} \cdot \vec{y} = 1 \text{ and } \vec{y} \cdot \vec{z} = 3\sqrt{3}.$$

(a) What is the angle (in radians) between $\vec{x}$ and $\vec{y}$?

(b) What are the possible values of $\vec{x} \cdot \vec{z}$?