SOLUTIONS TO PRELIM 2 MATH 221 FALL 2007

Problem 1

It is easy to find:

$$\begin{cases} T(1) = 0 \\ T(t) = 1 \\ T(t^2) = 2t + 2 \end{cases}$$

So the matrix would be:

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Surely its determinant is 0.

Problem 2

Isomorphisms are dimension-invariant, so there is no isomorphism from a 10-dim space (P_9) to a 9-dim space ($R^{3\times 3}$).

Problem 3

The best way to test orthogonality is by checking whether $A^T A = I_n$. Easy to see A is orthogonal and B isn't.

Problem 4

Let v', w' be the normalization of v, w (i.e. to make them unit vectors). Then put $A = [v', w']$ and orthogonal projection is given by AA^T, which equals:

$$\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then the first column represents the image of x in the orthogonal projection and hence is the best approximation of x in the least square sense.

Problem 5

$$||\cos t - \sin t|| = \sqrt{\langle \cos t - \sin t, \cos t - \sin t \rangle} = \sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} (\cos t - \sin t)^2 dt} = \sqrt{2}$$

Problem 6

By Gram-Schmidt Process it is not difficult to find:

$$Q = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$R = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

and the area of the parallelogram is given by $\sqrt{\det A^T A} = 2$

Problem 7

First by subtracting the $(n-1)$-th row from the n-th row, we can get $(0, 0, ..., 1)$ in the last row. Iterate this process to subtract $(i-1)$-th row from the i-th row in reverse order we can get a matrix like this:

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Surely it has determinant 1.