Math 2210 **Fall 2009**

First Prelim-Solutions

October 1, 2009

These solutions contain **more than** just the answers to the problems. They also contain comments about the problems and, sometimes, multiple methods for obtaining answers.

1. (*15 points*) Find, if possible, a number k such that the two vectors

$$\begin{bmatrix} 1-k \\ 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

are linearly dependent, or show that there is no such number.

Here are two solutions. *Solution 1:* Since neither vector is equal to the zero vector, it is equivalent to determine for what k, if any, the vectors, as listed, are redundant, i.e., for what k is the second vector a multiple of the first:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = c \begin{bmatrix} 1-k \\ 2 \\ -1 \end{bmatrix},$$

for some scalar c. Comparing the second coordinates, we see that, whatever the value of k, $2c = 2$, i.e., $c = 1$. Similarly, comparing third coordinates, we see that $-c = 3$, contradicting $c = 1$. So, no value of k will make the two vectors linearly dependent.

Solution 2: This method uses row-reduction, applied to the matrix A obtained from the two given column vectors:

$$A = \begin{bmatrix} 1-k & 1 \\ 2 & 2 \\ -1 & 3 \end{bmatrix}.$$

This row-reduction is easy, and we leave this to the student. The result is that no matter what the value of k, rrefA is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This shows that there are no free variables in the solution of $A\vec{x} = \vec{0}$, and so the columns of A do not admit a non-trivial relation, i.e., they are linearly independent whatever the value of k.

2. (*25 points*) Consider the transformation $T(\vec{x}) = x_1 - 5x_2 + 2x_3$ from $\mathbb{R}^3$ to $\mathbb{R}^1$.

a. *(10 pts)* Show that it is a linear transformation, and find its matrix (that is, a matrix A of the appropriate size such that $T(\vec{x}) = A\vec{x}$).

Solution 1: It is easy to verify that the $1x3$ matrix $A = [1 \ -5 \ 2]$ satisfies $T(\vec{x}) = x_1 - 5x_2 + 2x_3 = A\vec{x}$). Therefore, by the textbook definition, T is a linear transformation, and its matrix is $[1 \ -5 \ 2]$.

Solution 2: For arbitrary 3x1 column vectors $\vec{x}$ and $\vec{y}$, together with arbitrary scalar k, verify that $T(\vec{x}+\vec{y}) = T(\vec{x})+T(\vec{y})$ and $T(k\vec{x}) = kT(\vec{x})$. This is a matter of following the definition of T that is given and then performing some elementary algebra. Most students who used this method of solution did this correctly, and the computation won't be repeated here.

Two repeated errors appeared in solutions to this problem. These were not penalized by point deductions, but they will be if they re-appear on the next prelim. First, it was asserted that to show that T is a linear transformation, one has to show that "T is *closed* with respect to vector addition and scalar multiplication." This is incorrect terminology; it applies to the properties of a *linear subspace* not the properties of a *linear transformation.* Second, in a similar vein, it was asserted that one had to verify that $T(\vec{0}) = \vec{0}$. This is superfluous, since it follows from the property $T(k\vec{x}) = kT(\vec{x})$. There is a related property that one does need to verify in the case of linear subspaces: namely, that the subspace contains the zero vector.

Please keep the concept of a linear subspace and a linear transformation separate.

b. *(15 pts)* Let

$$\vec{b_1} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{b_2} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

Show that they form a basis for $ker(T)$, the nullspace of T.

Solution: To show that the vectors form a basis of the linear subspace $ker(T)$, it must be verified that they are linearly independent and that they span $ker(T)$.

The easiest way to show that the vectors are linearly independent is to show that neither is a multiple of the other. $\vec{b_1}$ cannot be a multiple of $\vec{b_2}$ because the latter has 0 as its second coordinate, whereas the former has 1. A similar observation applied to the third coordinates shows that $\vec{b_2}$ cannot be a multiple of $\vec{b_1}$. Another method for verifying linear independence is to place the vectors side-by-side in a 3x2 matrix

$$B = \begin{bmatrix} 5 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and then to row-reduce B, showing that $rref(B)$ equals

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

As stated in the solution for Problem 1, this shows that there are no free variables for the solution to $B\vec{x} = \vec{0}$, hence that there is only the zero solution. This means that the columns of B do not admit a non-trivial relation, i.e., they are linearly independent.

It remains to show that the vectors $\vec{b_1}$ and $\vec{b_2}$ span $ker(T)$. This is verified by solving the equation $A\vec{x} = \vec{0}$. Since A is already in row-reduced echelon form, one can write down the solution directly: $x_1 = 5x_2 - 2x_3$. The free variables are x_2 and x_3. Set the former equal to s and the latter equal to t. Then, the general solution can be written as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = s\vec{b_1} + t\vec{b_2}.$$

This shows that $\vec{b_1}$ and $\vec{b_2}$ span $ker(T)$.

Many students failed to show that the vectors span $ker(T)$. By using material from a later section—which, technically, is not within the scope of the prelim— it is possible to deduce indirectly that the vectors span $ker(T)$, but some argument is needed for this in order to get credit.

3. (*20 points*) Let A be the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 3 & -2 & 19 \end{bmatrix}.$$

With

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

consider the system of equations

$$(*) \qquad\qquad A\vec{x} = \vec{b}.$$

a.(*6 pts*) Let $\vec{b} = \begin{bmatrix} 8 \\ -3 \\ 0 \end{bmatrix}$. Show that the system is consistent and find the general solution.

Solution: Form the augmented matrix $[A|\vec{b}]$ and row reduce:

$$[A|\vec{b}] = \begin{bmatrix} 1 & 2 & 1 & 8 \\ 0 & -1 & 2 & -3 \\ 3 & -2 & 19 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -2 & 3 \\ 0 & -0 & 0 & 0 \end{bmatrix}.$$

x_1 and x_2 are the pivot variables (leading variables); x_3 is the free variable. Introduce the parameter $x_3 = t$. Then the general solution is

$$\left\{ \begin{array}{ccc} x_1 & = & 2-5t \\ x_2 & = & 3+2t \\ x_3 & = & t \end{array} \right\}.$$

This is a solution for any $t \in \mathbb{R}$; in particular, the system is consistent. In vector form (not required), the solution is

$$\vec{x} = \begin{bmatrix} 2-5t \\ 3+2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}.$$

b. *(14 pts)* Find real numbers s and t such that the condition

$$b_3 = sb_1 + tb_2$$

is a necessary and sufficient condition that the system (*) will have any solution.

Solution: Let $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, form the augmented matrix $[A|\vec{b}]$, and row-reduce:

$$[A|\vec{b}] = \begin{bmatrix} 1 & 2 & 1 & b_1 \\ 0 & -1 & 2 & b_2 \\ 3 & -2 & 19 & b_3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 5 & b_1 + 2b_2 \\ 0 & -1 & 2 & -b_2 \\ 0 & 0 & 0 & -3b_1 - 8b_2 + b_3 \end{bmatrix}.$$

The system is consistent if and only if $-3b_1 - 8b_2 + b_3 = 0$, i.e., $b_3 = 3b_1 + 8b_2$. So, the answers are $s = 3$ and $t = 8$.

Common mistakes: (a) Row-reduce $[A|\vec{v}]$, where $\vec{v} = \begin{bmatrix} s \\ t \\ -1 \end{bmatrix}$: No credit.

(b) $rref[A|\vec{b}] = \begin{bmatrix} 1 & 0 & 5 & b_1 \\ 0 & -1 & 2 & -b_2 \\ 0 & 0 & 0 & b_3 \end{bmatrix}$: No credit.

(c)Partial answer $8s = 3t$ found by plugging in $\vec{b} = \begin{bmatrix} 8 \\ -3 \\ 0 \end{bmatrix}$: 2 pts.

4. *(20 points)* Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation and $S : \mathbb{R}^n \to \mathbb{R}^n$ an invertible linear transformation. Show that

$$ker(ST) = ket(T).$$

Solution: To show that two sets X and Y are equal, you must show that every element of X belongs to Y and that every element of Y belongs to X. We apply this to $X = ker(ST)$ and $Y = ker(T)$.

So, suppose $\vec{v} \in ker(ST)$. Then $S(T(\vec{v})) = \vec{0}$. Apply S^{-1} to this equation: $S^{-1}(S(T(\vec{v})) = S^{-1}(\vec{0})$. On the left-hand side, the S^{-1} and S cancel, leaving $T(\vec{v})$. On the right-hand side, since S^{-1} is a linear transformation, we get $S^{-1}(\vec{0}) = \vec{0}$. Therefore, the equation becomes, $T(\vec{v}) = \vec{0}$, which means $\vec{v} \in ker(T)$, as desired.

Next, suppose $\vec{v} \in ker(T)$, i.e., $T(\vec{v}) = \vec{0}$. Apply S to both sides, obtaining $S(T(\vec{v})) = S(\vec{0}) = \vec{0}$. This tells us that $\vec{v} \in ker(ST)$, as desired.

5. (*20 points*) Let

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We want to consider matrices A of the appropriate size, such that

$$(**) \qquad\qquad A\vec{v} = \vec{b}.$$

Furthermore, we only want to consider matrices A which are already in row-reduced echelon form.

Find, if possible two different such "rref" matrices, or show that there is only one, or none.

Solution: There are, in fact, more than two "rref" matrices A of the type asked for. Just for amusement, we'll present a list of all possible such matrices.

First, to reduce the search, notice that we can eliminate the zero matrix, since the image vector $\vec{b}$ is not zero. So, the desired matrices must have one or two leading entries. Next, notice also that the second row of A cannot be [0 0 1], since this would force $A\vec{v} = \vec{b}$ to have its second entry equal to 3(by definition of matrix multiplication), contradicting what is given for $\vec{b}$. This leaves the following possibilities for A:

$$\begin{bmatrix} 1 & a & b \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & a \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix},$$

where a and b are some as yet undetermined real numbers. To determine a and b, use each of the possible candidates for A in the equation $A\vec{v} = \vec{b}$. For the first matrix, this gives the equation $1 + 2a + 3b = 1$, i.e., $a = -3b/2$. For the second matrix, we get $a = -1/3$. For the third, we get $3 = 1$, which is impossible. So the third matrix is rejected. For the fourth matrix, we get $1 + 3a = 1$, i.e., $a = 0$, and $2 + 3b = 0$, i.e., $b = -2/3$. Therefore, the following matrices give a complete list of those satisfying the desired conditions:

$$\begin{bmatrix} 1 & -3b/2 & b \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2/3 \end{bmatrix},$$

where b can be any real number.