Homework 5
Solutions

Chapter 3.3

Problem 10

The first column is redundant, and $\begin{bmatrix}1\\0\\0\end{bmatrix}$ is in the kernel. No other columns are redundant, so basis for the kernel is $\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$ and a basis for the image is $\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}\right)$.

Problem 16

The third column is redundant, since $3\vec{v}_1 + 2\vec{v}_2 - 1\vec{v}_3 + 0\vec{v}_4 = 0$. The fourth column is not redundnant. Thus a basis for the kernel is $\left(\begin{bmatrix}3\\2\\-1\\0\end{bmatrix}\right)$, while $\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\\4\end{bmatrix}\right)$.

Problem 22

$$rref\begin{bmatrix}2 & 4 & 8\\4 & 50 & 1\\7 & 9 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & -6\\0 & 1 & 5\\0 & 0 & 0\end{bmatrix}$$

The third vector is the only redundant vector. Since $-6\vec{v}_1 + 5\vec{v}_2 - 1\vec{v}_3 = \vec{0}$, the vector $\begin{bmatrix}-6\\5\\-1\end{bmatrix}$ is in the kernel, and is in fact the basis of the kernel. The basis of the image is $\left(\begin{bmatrix}2\\4\\7\end{bmatrix}, \begin{bmatrix}4\\5\\9\end{bmatrix}\right)$.

Problem 30

After doing row-reduction, and choosing free variables, we get that the basis is $\left(\begin{bmatrix}1\\2\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\0\\1\end{bmatrix}\right)$.

Problem 38

a The rank of a 3×5 matrix is $0, 1, 2$ or 3, so $dim(ker) = 5 - rank \in \{2, 3, 4, 5\}$

b The rank of a 7×4 matrix is $0, 1, 2, 3$ or 4, so $dim(im) = rank \in \{0, 1, 2, 3, 4\}$

Chapter 3.4

Problem 8

The system is inconsistent, so $\vec{x}$ is not in the span of $\vec{v}_1$ and $\vec{v}_2$.

Problem 18

The system is inconsistent, so $\vec{x}$ is not in the span of $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$.

Problem 20

a $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, with inverse $S^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Then $B = S^{-1}AS = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

b We use the commutative diagram:

$$\begin{array}{ccc} \vec{x} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} & \xrightarrow[T]{} & T(\vec{x}) = A\vec{x} = c_1 A\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \downarrow & & \downarrow \\ [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} & \xrightarrow[T]{} & [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 2c_1 \\ 0 \end{bmatrix} \end{array}$$

So $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2c_1 \\ 0 \end{bmatrix}$ and we find $B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

c $B = [[T(\vec{v}_1)]_{\mathcal{B}}[T(\vec{v}_2)]_{\mathcal{B}}] = \left[\left[\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]_{\mathcal{B}}\left[\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right]_{\mathcal{B}}\right] = \left[\begin{bmatrix} 2 \\ 2 \end{bmatrix}_{\mathcal{B}}\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\mathcal{B}}\right] = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$. □

Problem 44

$\vec{x} = 2\vec{v}_1 + (-1)\vec{v}_2 = 2\begin{bmatrix} 8 \\ 4 \\ -1 \end{bmatrix} + (-1)\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ 6 \\ -1 \end{bmatrix}$

Problem 56

If $S = [\vec{v}_1\vec{v}_2]$, where the column vectors are the desired basis, then, by Fact 3.4.1, $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = S\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix} = S\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, hence $S\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

Hence $S = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 12 & -7 \\ 14 & -8 \end{bmatrix}$. The basis is $\left\{\begin{bmatrix} 12 \\ 14 \end{bmatrix}, \begin{bmatrix} -7 \\ -8 \end{bmatrix}\right\}$.