Homework 4
Solutions

Chapter 3.1

Problem 6

$$\begin{bmatrix} 1 & 1 & 1 & \vdots & 0 \\ 1 & 1 & 1 & \vdots & 0 \\ 1 & 1 & 1 & \vdots & 0 \end{bmatrix} \xrightarrow[R_2-R_1]{R_3-R_1} \begin{bmatrix} 1 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

So we have that $x_1 + x_2 + x_3 = 0$, so we have:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -r-t \\ r \\ t \vdots \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence

$$ker(A) = span \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Problem 22

By Fact 3.1.3, $Im(A) = span \left\{ \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix} \right\}$.

Since

$$\begin{bmatrix} 2 & 1 & \vdots & 3 \\ 3 & 4 & \vdots & 2 \\ 6 & 5 & \vdots & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & \vdots & 2 \\ 0 & 1 & \vdots & -1 \\ 0 & 0 & \vdots & 0 \end{bmatrix}$$

This computation shows that the third column is a linear combination of the first two. Only the first two vectors are independent, and the image is thus a plane.

Problem 38

a If a vector $\vec{x}$ is in $ker(A^k)$, then $A^k\vec{x} = \vec{0}$. But $A^{k+1}\vec{x} = AA^k\vec{x} = A\vec{0} = \vec{0}$, so $\vec{x}$ is also in $ker(A^{k+1})$. Therefore $ker(A) \subseteq ker(A^2) \subseteq ker(A^3) \subseteq ker(A^4) \subseteq \dots$

The kernels need not be equal - see exercise 37 for an example.

b If a vector $\vec{y}$ is in $Im(A^{k+1})$ then $\vec{y} = A^{k+1}\vec{x} = A^k(A\vec{x}) = A^k\vec{z}$, so $\vec{y}$ is in $Im(A^k)$. Therefore we have that $Im(A) \supseteq Im(A) \supseteq Im(A^2) \supseteq Im(A^3) \supseteq \dots$.

The images need not be equal - see exercise 37 for an example.

Chapter 3.2

Problem 2

W is not a subspace, since it contains the vector $\vec{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, but not $(-1)\vec{v} = \begin{bmatrix}-1\\-2\\-3\end{bmatrix}$.

Problem 4

By Fact 3.2.2, $span(\vec{v_1}, \ldots, \vec{v_m}) = Im[\vec{v}_1 \ldots \vec{v}_m]$, which is always a subspace. One can also prove this by checking the definition of a subspace.

Problem 16

$\begin{bmatrix}3\\2\\1\end{bmatrix}$ is not a multiple of $\begin{bmatrix}1\\1\\1\end{bmatrix}$, so it is not redundant. But $\begin{bmatrix}6\\5\\4\end{bmatrix} = 3\begin{bmatrix}1\\1\\1\end{bmatrix} + 1\begin{bmatrix}3\\2\\1\end{bmatrix}$, so it is redundant. Thus the vectors are linearly dependent.

Problem 34

$$A\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4\end{bmatrix}\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \vec{0}$$

$$\vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3 + 4\vec{v}_4 = \vec{0} \Rightarrow \vec{v}_4 = -\frac{1}{4}\vec{v}_1 - \frac{1}{2}\vec{v}_2 - \frac{3}{4}\vec{v}_3$$