Homework 2
Solutions

Chapter 1.3

Problem 4

Since $rref\begin{bmatrix}1 & 4 & 7\\ 2 & 5 & 8\\ 3 & 6 & 9\end{bmatrix} = \begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 2\\ 0 & 0 & 0\end{bmatrix}$, the rank is 2. $\square$

Problem 26

Since the system has a unique solution, then $rank(A) = 3$, so $rref(A) = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$. Since all the variables are leading, the system $A\vec{x} = \vec{c}$ cannot have infinitely many solutions, so it could have a unique solution (for example, if $\vec{c} = \vec{b}$) or no solution at all. $\square$

Problem 44

Let $E = rref(A)$ and note that all the entries in the last row of E must be 0, by the definition of $rref$. If $\vec{c}$ is any vector in $\mathbb{R}^n$ whose last component is not 0, then the system $E\vec{x} = \vec{c}$ will be clearly inconsistent.

Now consider the elementary row operations that transform A into E, and apply the opposite operations, in reverse order, to the augmented matrix $\begin{bmatrix}E & \vdots & \vec{c}\end{bmatrix}$. We will obtain the augmented matrix $\begin{bmatrix}A & \vdots & \vec{b}\end{bmatrix}$ that represents an inconsistent system $A\vec{x} = \vec{b}$, as required. $\square$

Chapter 2.1

Problem 6

We note that $x_1\begin{bmatrix}1\\2\\3\end{bmatrix} + x_2\begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}1 & 4\\ 2 & 5\\ 3 & 6\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$, so T is indeed linear, with matrix $\begin{bmatrix}1 & 4\\ 2 & 5\\ 3 & 6\end{bmatrix}$. $\square$

Problem 13

1. First suppose $a \neq 0$. We attempt to solve the equation $\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$ for x_1 and x_2.

$$\left|\begin{matrix}ax_1 + bx_2 & = & y_1\\ cx_1 + dx_2 & = & y_2\end{matrix}\right| \xrightarrow{R_1/a} \left|\begin{matrix}x_1 + \frac{b}{a}x_2 & = & \frac{1}{a}y_1\\ cx_1 + dx_2 & = & y_2\end{matrix}\right| \xrightarrow{R_2 \to R_2 - cR_1} \left|\begin{matrix}x_1 + \frac{b}{a}x_2 & = & \frac{1}{a}y_1\\ \frac{ad-bc}{a}x_2 & = & fraccay_1 + y_2\end{matrix}\right|$$

We can solve this system for x_1 and x_2 of and only if $ad - bc \neq 0$, as claimed.

If $a = 0$ then the system becomes:

$$\left|\begin{matrix}bx_2 & = & y_1\\ cx_1 + dx_2 & = & y_2\end{matrix}\right| \xrightarrow{R_1 \leftrightarrow R_2} \left|\begin{matrix}cx_1 + dx_2 & = & y_2\\ bx_2 & = & y_1\end{matrix}\right|$$

We can solve fot x_1 and x_2 provided that both b and c are nonzeroe, i.e. $bc \neq 0$. Since $a = 0$, this means that $ad - bc \neq 0$, as claimed.

2. First assume that $D := ad - bc \neq 0$ and $a \neq 0$. We continue solving the system above and we get:

$$\left|\begin{matrix}x_1 + \frac{b}{a}x_2 & = & \frac{1}{a}y_1\\ \frac{D}{a}x_2 & = & -\frac{c}{a}y_1 + y_2\end{matrix}\right| \xrightarrow{R_2 \cdot \frac{a}{D}} \left|\begin{matrix}x_1 + \frac{b}{a}x_2 & = & \frac{1}{a}y_1\\ x_2 & = & -\frac{c}{D}y_1 + \frac{a}{D}y_2\end{matrix}\right| \xrightarrow{R_1 - \frac{b}{a}R_2} \left|\begin{matrix}x_1 & = & \overbrace{(\frac{1}{a} + \frac{bc}{aD})}^{\frac{d}{D}}y_1 - \frac{b}{D}y_2\\ x_2 & = & -\frac{c}{D}y_1 + \frac{a}{D}y_2\end{matrix}\right|$$

It follows that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ as claimed.

If $D := ad - bc \neq 0$ and $a = 0$, the system becomes:

$$\left| \begin{matrix} cx_1 + dx_2 & = & y_2 \\ bx_2 & = & y_1 \end{matrix} \right| \xrightarrow[R_2/b]{R_1/c} \left| \begin{matrix} x_1 + \frac{d}{c}x_2 & = & \frac{1}{c}y_2 \\ x_2 & = & \frac{1}{b}y_1 \end{matrix} \right| \xrightarrow{R_1 \to R_1 - \frac{d}{c}R_2} \left| \begin{matrix} x_1 & = & -\frac{d}{bc}y_1 + \frac{1}{c}y_2 \\ x_2 & = & \frac{1}{b}y_1 \end{matrix} \right|$$

It follows that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{d}{bc} & \frac{1}{c} \\ \frac{1}{b} & 0 \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ as claimed. □

Problem 44

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_2x_3 - v_3x_2 \\ v_3x_1 - v_1x_3 \\ v_1x_2 - v_2x_1 \end{bmatrix} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

So T is a linear map, with matrix $\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$. □

Chapter 2.2

Problem 2

By Fact 2.2.3, this matrix is $\begin{bmatrix} \cos(60°) & -\sin(60°) \\ \sin(60°) & \cos(60°) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$. □

Problem 24

1. $A = [\vec{v} \quad \vec{w}]$ so $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{v}$ and $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{w}$.

   Since A preserves lenght, both $\vec{v}$ and $\vec{w}$ must be unit vectors. Also, since it preserves angles, and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are clearly perpendicular, then $\vec{v}$ and $\vec{w}$ must also be perpendicular.

2. Since $\vec{w}$ si a unit vector perpendicular to $\vec{v}$, it can be obtained by rotating $\vec{v}$ through 90°, either in the counterclockwise or in the clockwise direction. Using the corresponding rotation matrices, we see that $\vec{w} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} -b \\ a \end{bmatrix}$ or $\vec{w} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} b \\ -a \end{bmatrix}$.

3. Using the above, A is either of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, representing a rotation, or $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, representing a reflection. □

Problem 38

1. $A = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$ so $\det(A) = u_1^2u_2^2 - u_1u_2u_1u_2 = 0$.

2. $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ so $\det(A) = -a^2 - b^2 = -(a^2 + b^2) = -1$.

3. $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ so $\det(A) = -a^2 - (-b^2) = a^2 + b^2 = 1$.

4. $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, both having determinant $\det(A) = 1^2 - 0 = 1$. □