Math 221

Final Exam

December 9, 2004

This is a closed book exam. Calculators are not allowed. Show all work!

1) Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be given by $T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 6 \end{bmatrix} \vec{x}$.

a) (8 points) Find a basis for the kernel of T.

b) (7 points) Find a basis for the image of T.

2) Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \vec{x}$. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ be an ordered basis of $\mathbb{R}^2$.

a) (5 points) Give the coordinates of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ with respect to $\mathcal{B}$.

b) (5 points) Give the coordinates of $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$ with respect to $\mathcal{B}$.

c) (5 points) Find the matrix of T with respect to $\mathcal{B}$.

3) *For this problem you need not show your work.*

a) (4 points) Give an example of a 3×3 orthogonal matrix that is *not* the identity matrix.

b) (4 points) Give an example of a 4×5 matrix in row reduced echelon form that has exactly three leading ones.

c) (4 points) Give an example of a 3×3 matrix whose trace is 9 and determinant is 7.

d) (3 points) Give an example of a matrix A which has 3 as eigenvalue with algebraic multiplicity two and geometric multiplicity one.

4) Let A and B be 3×3 matrices satisfying $\det(A) = -3$ and $\det(B) = 2$.

a) (5 points) Find $\det(AB)$.

b) (5 points) Find $\det(3B^{-1}A^T)$.

c) (5 points) Give explicit A and B (with specified determinants above) such that $\det(A - B) = 0$ or show no such A and B exist.

5) Let $A = \begin{bmatrix} 1 & -2 \\ 3 & 6 \end{bmatrix}$.

a) (8 points) Find a matrix S and a diagonal matrix D such that $A = SDS^{-1}$.

b) (7 points) For positive integers n, find a formula for A^n as a product of three matrices.

6) Let $P : \mathbb{R}^n \to \mathbb{R}^n$ satisfy $P^2 = P$.

a) (5 points) Show the eigenvalues λ of P satisfy $\lambda^2 = \lambda$. Find all possible eigenvalues of P.

b) (5 points) Show that the kernel of P and image of P are eigenspaces of P.

c) (5 points) Using b), show that P has a basis of eigenvectors.