Math 2210 Linear Algebra: Prelim 2, Solutions

October 29, 2009

Problem 1: (Total= 20 pts) Let $\mathcal{P}_n$ be the vector space of all polynomials $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ of degree $\leq n$. Let $D : \mathcal{P}_n \to \mathcal{P}_n$ be the differentiation operator

$$D(p(x)) = \frac{d}{dx}(p(x)).$$

Let $\mathcal{B} = \{1, x, x^2, \ldots, x^n\}$ be the basis of $\mathcal{P}_n$ consisting of all the powers of x up to x^n.

a. (5 pts) Check that D is a linear transformation.

b. (5 pts) Calculate the matrix of D with respect to the basis $\mathcal{B}$.

c. (5 pts) What is the rank of D?

d. (5 pts) Is D an isomorphism? (Give reasons.)

Solution:

a. We must show, for any polynomials $p(x), q(x)$ in $\mathcal{P}_n$ and any real number k, that $D(p(x) + q(x)) = D(p(x)) + D(q(x))$ and $D(kp(x)) = kD(p(x))$. But these equations follow immediately from well-known properties of differentiation. We omit further details.

b. The equation $D(x^k) = kx^{k-1}$, valid for all integers $k \geq 0$, shows how to express the vectors $D(x^k)$ in terms of the basis vectors of $\mathcal{B}$. Since the monomial x^{k-1} has coordinate vector $[x^{k-1}]_{\mathcal{B}} = \vec{e}_k$, the k^{th} standard basis vector of $\mathbb{R}^{n+1}$, we have that, for each k, the coordinate vector $[D(x^k)]_{\mathcal{B}}$ is the column vector $k\vec{e}_k$. Therefore, the desired matrix M, when written in terms of its columns, is given by

$$M = [\vec{0}, \vec{e_1}, 2\vec{e_2}, \ldots, n\vec{e_n}].$$

c. $rank(D) = rank(M) = \#$ of independent cols. of $M = n$.

d. D is <u>not</u> an isomorphism. If it were, then M would not have a zero column (which implies that the kernel of D is not trivial), and, furthermore, the rank of D would be equal to $n + 1$.

Problem 2: (Total =15 pts) Again, let $\mathcal{P}_n$ be the vector space of all polynomials $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ of degree $\leq n$. A polynomial $p(x) \in \mathcal{P}_n$ is called *even* if $p(x) = p(-x)$. Let E be the subset of $\mathcal{P}_n$ consisting of all even polynomials.

a. (5 pts) Prove that E is a linear subspace of $\mathcal{P}_n$.

b. (5 pts) Find a basis of E and give reasons why it is a basis.

c. (5 pts) Compute the dimension of E. (Consider separately the cases $n = 2m$ and $n = 2m+1$, i.e., n even and odd, respectively.)

Solution:

a. The constantly zero polynomial has the same value at any x and $-x$, namely, the value 0, so it is even, i.e., $0 \in E$. If polynomials $p(x) = p(-x)$ and $q(x) = q(-x)$, then $p(x) + q(x) = p(-x) + q(-x)$, and $kp(x) = kp(-x)$, for any real k. So E is closed under addition and scalar multiplication of polynomials, verifying that it is a linear subspace of $\mathcal{P}_n$.

b. Suppose that $2m$ is the greatest even integer $\leq n$. Then we assert that the set of monomials $\mathcal{A} = \{1, x^2, x^4, \ldots, x^{2m}\}$ is a basis of E. To justify this assertion, we note that $\mathcal{A}$ is linearly independent, since it is a subset of the basis $\mathcal{B}$ given in the previous problem. It remains to show that $\mathcal{A}$ spans E. So, choose any polynomial $p(x) \in E$. We may write it as

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n,$$

because it is in $\mathcal{P}_n$. But, since $p(x) = p(-x)$ and since $(-x)^k = -x^k$ when k is odd, we have

$$p(x) = a_0 - a_1 x + a_2 x^2 - \ldots + (-1)^n a_n x^n.$$

Comparing coefficients, we see that $a_k = -a_k$, for every odd k, which means that each such $a_k = 0$. Thus, the odd-degree terms of $p(x)$ vanish, and we are left with

$$p(x) = a_0 + a_2 x^2 + \ldots + a_{2m} x^{2m}.$$

This shows that $p(x)$ is a linear combination of the elements of $\mathcal{A}$, i.e., that $\mathcal{A}$ spans E, completing the verification that $\mathcal{A}$ is a basis of E.

c. In either case, the dimension of E is $m + 1$, the number of elements in the basis described in part *b*).

Problem 3: (20 pts) Suppose that A is an upper -triangular matrix, i.e.,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix},$$

and that its columns form an orthonormal basis of $\mathbb{R}^3$. Prove that

$$A = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix},$$

i.e., A has each diagonal entry equal to 1 or -1 and each off-diagonal entry equal to 0.

Solution: The columns of A are orthonormal. We start by taking the dot product of the first column with each of the three columns. This yields, first, going from left to right, $a_{11}^2 = 1$, so $a_{11} = \pm 1$. Substitute ± 1 for a_{11}. Second, taking the dot product of the first and second column, we get $(\pm 1)a_{12} = 0$, so $a_{12} = 0$. Substitute 0 for a_{12}. Third, $(\pm 1)a_{13} = 0$, so $a_{13} = 0$. Substitute 0 for a_{13}. Next, take the dot product of the second column with the second and third columns. This yields, first, $a_{22}^2 = 1$, so $a_{22} = \pm 1$. Again do the indicated substitution. Second, $(\pm 1)a_{23} = 0$, so $a_{23} = 0$, and substitute again. Finally, take the dot product of the last column with itself. This yields $a_{33}^2 = 1$, so $a_{33} = \pm 1$. This completes the calculation that A has the desired form.

Problem 4: (20 pts) Suppose that the basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ of $\mathbb{R}^2$ is given by

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(You are <u>not</u> required to show that $\mathcal{B}$ is a basis.) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation whose matrix with respect to $\mathcal{B}$ is

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Let $\mathcal{A} = \{\vec{e}_1, \vec{e}_2\}$ be the standard basis of $\mathbb{R}^2$. Calculate the matrix of T with respect to the basis $\mathcal{A}$.

Solution: Our goal is to evaluate $T(\vec{e}_1)$ and $T(\vec{e}_2)$ in terms of $\vec{e}_1$ and $\vec{e}_2$. We'll proceed by first evaluating $T(\vec{v}_1)$ and $T(\vec{v}_2)$ using the information in the given matrix:

$$T(\vec{v}_1) = 0\vec{v}_1 + 1\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad T(\vec{v}_2) = \vec{v}_1 + 2\vec{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Next, express $\vec{e}_1$ and $\vec{e}_2$ in terms of $\vec{v}_1$ and $\vec{v}_2$: $\vec{e}_1 = -\vec{v}_1 + \vec{v}_2$, and $\vec{e}_2 = \vec{v}_1$. We can then evaluate

$$T(\vec{e}_1) = T(-\vec{v}_1) + T(\vec{v}_2) = -\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

and

$$T(\vec{e}_2) = T(\vec{v}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

These computations give the two columns of the desired matrix, which is:

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.$$

An alternative solution to the problem can be obtained by first computing the basis change matrix S, which expresses the basis $\mathcal{B}$ in terms of the standard basis $\mathcal{A}$ and then writing the coefficients as the columns of S. But, this is easy, because the vectors $\vec{v}_1$ and $\vec{v}_2$ of $\mathcal{B}$ are written as columns in $\mathbb{R}^2$, which immediately gives the desired columns. So,

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Next, compute the inverse S^{-1}, which can be done from the formula in the textbook: If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $D \neq 0$ denotes the determinant of M, namely $D = ad - bc$, then $M^{-1} = D^{-1} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.From this, we get

$$S^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then, the desired matrix is given by

$$S \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} S^{-1}.$$

If one multiplies this out, using the matrices computed for S and for S^{-1}, one gets the same answer as was obtained by the first method.

Problem 5: (Total = 25 pts) Consider the basis $\{\vec{v}_1, \vec{v}_2\}$ of $\mathbb{R}^2$, where

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

a. (15 pts) Apply the Gram-Schmidt process to $\{\vec{v}_1, \vec{v}_2\}$ to produce an orthonormal basis $\{\vec{u}_1, \vec{u}_2\}$. (We are interested not only in $\{\vec{u}_1, \vec{u}_2\}$ but also in whether you are doing the Gram-Schmidt process correctly. So, display your equations neatly and keep careful track of the expressions of $\vec{u}_1$ and $\vec{u}_2$ in terms of $\vec{v}_1$ and $\vec{v}_2$. You are <u>not</u> required to show that $\{\vec{v}_1, \vec{v}_2\}$ or $\{\vec{u}_1, \vec{u}_2\}$ are bases of $\mathbb{R}^2$.)

b. (10 pts) Use (a) to obtain a QR factorization of the matrix

$$[\vec{v}_1, \vec{v}_2] = \begin{bmatrix} 0 & 3 \\ 1 & 4 \end{bmatrix}.$$

Solutions:

a. $\vec{v}_1$ is a unit vector, so

$$\vec{u}_1 = \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Next, we have

$$\vec{u}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1.$$

Therefore,

$$\vec{u}_2^{\perp} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} - 4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

So, normalizing $\vec{u}_2^{\perp}$, we get

$$\vec{u}_2 = \frac{1}{3}\vec{u}_2^{\perp} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Therefore, the orthonormal basis obtained from $\{\vec{v}_1, \vec{v}_2\}$ by the Gram-Schmidt process is

$$\{\vec{u}_1, \vec{u}_2\} = \{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\}.$$

Notice that this basis is the standard basis of $\mathbb{R}^2$ written in the reverse order. In general, the Gram-Schmidt process starts with a basis which is *in a particular order*, and it ends with an orthonormal basis which is produced in a particular order. If you change the order of the initial basis, you will, in general, end with an orthonormal basis that is very different from the one you obtained originally. It will not be, in general, simply a re-ordering of this last. (Try it with the basis given in this problem but written in the reverse order.)

b. For the QR factorization: $Q = [\vec{u}_1, \vec{u}_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The matrix R is the basis-change matrix. The easiest way to calculate R is to express the basis $\{\vec{v}_1, \vec{v}_2\}$ in terms of $\{\vec{u}_1, \vec{u}_2\}$:

$$\vec{v}_1 = \vec{u}_1, \qquad \vec{v}_2 = 4\vec{u}_1 + 3\vec{u}_2.$$

This last equation follows easily by inspection. Then, take the coefficients in the expressions on the right-hand side, and use them as the columns of the desired 2×2 matrix R. So $R = \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix}$. Therefore, the QR factorization of $[\vec{v}_1, \vec{v}_2]$ is:

$$\begin{bmatrix} 0 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix}.$$