Math 2130 Workshop: The Laplace and Wave Equations

The *Laplace Equation* describes heat distribution in regions after they have settled down to a steady state (the heat distribution isn't changing over time). In this case, the temperature at any location inside the region is the average of the temperature in a sphere around that point. If the temperature in the region at position (x, y, z) is given by $f(x, y, z)$, then:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

Functions that satisfy this differential equation are called *harmonic*. Of course this holds if the whole region is a constant temperature; but there are other distributions that work as well (imagine a room being heated from the outside).

For each problem in this workshop, try to work out an intuition for why each function represents what they do.

1) Show that $f(x, y, z) = 2x + 2y$ is harmonic (a room being heated from the northeast corner and losing heat out the southwest corner).

> $f_x = 2$, so $f_{xx} = 0$. Similarly, $f_y = 2$ and $f_{yy} = 0$. Meanwhile $f_z = 0$ and so $f_{zz} = 0$. Their sum is 0.

2) Show that $f(x, y, z) = x^2 - y^2$ is harmonic (a room being heated from the east and west and losing heat from the north and south).

> $f_x = 2x$, so $f_{xx} = 2$. Similarly, $f_y = -2y$, so $f_{yy} = -2$. As before, $f_{zz} = 0$. Their sum is 0.

3) Show that $f(x, y, z) = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$ is harmonic except at the origin (a room being heated from a heat source at the center).

> $f_x = 2x(-\frac{1}{2})(x^2 + y^2 + z^2)^{-\frac{3}{2}} = -x(x^2 + y^2 + z^2)^{-\frac{3}{2}}$. So $f_{xx} = -2x^2(-\frac{3}{2})(x^2 + y^2 + z^2)^{-\frac{5}{2}} - (x2 + y^2 + z^2)^{-\frac{3}{2}}$. By symmetry, we have:
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> $$f_{xx} = 3x^2(x^2 + y^2 + z^2)^{-\frac{5}{2}} - (x^2 + y^2 + z^2)^{-\frac{3}{2}}$$
> $$f_{yy} = 3y^2(x^2 + y^2 + z^2)^{-\frac{5}{2}} - (x^2 + y^2 + z^2)^{-\frac{3}{2}}$$
> $$f_{zz} = 3z^2(x^2 + y^2 + z^2)^{-\frac{5}{2}} - (x^2 + y^2 + z^2)^{-\frac{3}{2}}$$
>
> Their sum is:
>
> $$3(x^2 + y^2 + z^2)(x^2 + y^2 + z^2)^{-\frac{5}{2}} - 3(x^2 + y^2 + z^2)^{-\frac{3}{2}} = 0.$$

4) Show that $f(x, y, z) = e^{-2y}\cos 2x$ is harmonic.

> $f_x = -2e^{-2y}\sin 2x$, and $f_{xx} = -4e^{-2y}\cos 2x$. Meanwhile $f_y = -2e^{-2y}\cos 2x$ and $f_{yy} = 4e^{-2y}\cos 2x$. As earlier, $f_{zz} = 0$. Their sum is 0.

The *Wave Equation* describes the propagation of waves: sound, light, and even waves on bodies of water. In the one-dimensional case (water waves in a trough or waves propagating down a string), if the intensity of the wave at position x and time t is given by $f(x,t)$, then:

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}.$$

5) Show that $f(x,t) = \sin(x + c\,t)$ (a sinusoidal wave moving to the left) satisfies the wave equation.

$f_x = \cos(x+ct)$ and $f_{xx} = -\sin(x+ct)$. Meanwhile $f_t = c\cos(x+ct)$ and $f_{tt} = -c^2\sin(x+ct)$.

6) Show that $f(x,t) = 5\sin(x + c\,t) - \cos(10\,x - 10\,c\,t)$ (a large sinusoidal wave moving to the left, with smaller, higher frequency waves moving to the right on top of them) satisfies the wave equation.

$f_x = 5\cos(x+ct) + 10\sin(10x-10ct)$ and $f_{xx} = -5\sin(x+ct) + 100\cos(10x-10ct)$. Meanwhile, $f_t = 5c\cos(x+ct) + 10c\sin(10x-10ct)$ and $f_{tt} = -5c^2\sin(x+ct) + 100c^2\cos(10x-10ct)$.

7) Show that $f(x,t) = \sin(c\,t)\cos(x)$ (a standing wave) satisfies the wave equation.

$f_x = -\sin(ct)\sin(x)$ and $f_{xx} = -\sin(ct)\cos(x)$. Meanwhile, $f_t = c\cos(ct)\cos(x)$ and $f_{tt} = -c^2\sin(ct)\cos(x)$.

8) Show that $f(x,t) = g(x + c\,t)$ (an arbitrary configuration moving to the left) satisfies the wave equation.

$f_x = g'(x+ct)$ and $f_{xx} = g''(x+ct)$. Meanwhile, $f_t = cg'(x+ct)$ and $f_{tt} = c^2g''(x+ct)$.