Math 2130 Workshop: The Laplace and Wave Equations

The Laplace Equation describes heat distribution in regions after they have settled down to a steady state (the heat distribution isn’t changing over time). In this case, the temperature at any location inside the region is the average of the temperature in a sphere around that point. If the temperature in the region at position \((x, y, z)\) is given by \(f(x, y, z)\), then:

\[
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.
\]

Functions that satisfy this differential equation are called harmonic. Of course this holds if the whole region is a constant temperature; but there are other distributions that work as well (imagine a room being heated from the outside).

For each problem in this workshop, try to work out an intuition for why each function represents what they do.

1) Show that \(f(x, y, z) = 2x + 2y\) is harmonic (a room being heated from the northeast corner and losing heat out the southwest corner).

\[f_x = 2, \text{ so } f_{xx} = 0. \text{ Similarly, } f_y = 2, \text{ and } f_{yy} = 0. \text{ Meanwhile } f_z = 0 \text{ and so } f_{zz} = 0. \text{ Their sum is 0.}\]

2) Show that \(f(x, y, z) = x^2 - y^2\) is harmonic (a room being heated from the east and west and losing heat from the north and south).

\[f_x = 2x, \text{ so } f_{xx} = 2. \text{ Similarly, } f_y = -2y, \text{ so } f_{yy} = -2. \text{ As before, } f_{zz} = 0. \text{ Their sum is 0.}\]

3) Show that \(f(x, y, z) = (x^2 + y^2 + z^2)^{-\frac{1}{2}}\) is harmonic except at the origin (a room being heated from a heat source at the center).

\[f_x = 2x(-\frac{1}{2})(x^2 + y^2 + z^2)^{-\frac{3}{2}} = -x(x^2 + y^2 + z^2)^{-\frac{3}{2}}. \text{ So } f_{xx} = -2x^2(-\frac{3}{2})(x^2 + y^2 + z^2)^{-\frac{5}{2}} - (x^2 + y^2 + z^2)^{-\frac{3}{2}}. \text{ By symmetry, we have:}\]

\[
\begin{align*}
    f_{xx} &= 3x^2(x^2 + y^2 + z^2)^{-\frac{5}{2}} - (x^2 + y^2 + z^2)^{-\frac{3}{2}} \\
    f_{yy} &= 3y^2(x^2 + y^2 + z^2)^{-\frac{5}{2}} - (x^2 + y^2 + z^2)^{-\frac{3}{2}} \\
    f_{zz} &= 3z^2(x^2 + y^2 + z^2)^{-\frac{5}{2}} - (x^2 + y^2 + z^2)^{-\frac{3}{2}}
\end{align*}
\]

Their sum is:

\[
3(x^2 + y^2 + z^2)(x^2 + y^2 + z^2)^{-\frac{5}{2}} - 3(x^2 + y^2 + z^2)^{-\frac{3}{2}} = 0.
\]

4) Show that \(f(x, y, z) = e^{-2y} \cos 2x\) is harmonic.

\[f_x = -2e^{-2y} \sin 2x, \text{ and } f_{xx} = -4e^{-2y} \cos 2x. \text{ Meanwhile } f_y = -2e^{-2y} \cos 2x, \text{ and } f_{yy} = 4e^{-2y} \cos 2x. \text{ As earlier, } f_{zz} = 0. \text{ Their sum is 0.}\]
The Wave Equation describes the propagation of waves: sound, light, and even waves on bodies of water. In the one-dimensional case (water waves in a trough or waves propagating down a string), if the intensity of the wave at position $x$ and time $t$ is given by $f(x, t)$, then:

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}.$$  

5) Show that $f(x, t) = \sin(x + ct)$ (a sinusoidal wave moving to the left) satisfies the wave equation.

$$f_x = \cos(x + ct) \text{ and } f_{xx} = -\sin(x + ct). \text{ Meanwhile } f_t = c \cos(x + ct) \text{ and } f_{tt} = -c^2 \sin(x + ct).$$

6) Show that $f(x, t) = 5 \sin(x + ct) - \cos(10x - 10ct)$ (a large sinusoidal wave moving to the left, with smaller, higher frequency waves moving to the right on top of them) satisfies the wave equation.

$$f_x = 5 \cos(x + ct) + 10 \sin(10x - 10ct) \text{ and } f_{xx} = -5 \sin(x + ct) + 100 \cos(10x - 10ct). \text{ Meanwhile, } f_t = 5c \cos(x + ct) + 10c \sin(10x - 10ct) \text{ and } f_{tt} = -5c^2 \sin(x + ct) + 100c^2 \cos(10x - 10ct).$$

7) Show that $f(x, t) = \sin(ct) \cos(x)$ (a standing wave) satisfies the wave equation.

$$f_x = -\sin(ct) \sin(x) \text{ and } f_{xx} = -\sin(ct) \cos(x). \text{ Meanwhile, } f_t = c \cos(ct) \cos(x) \text{ and } f_{tt} = -c^2 \sin(ct) \cos(x).$$

8) Show that $f(x, t) = g(x + ct)$ (an arbitrary configuration moving to the left) satisfies the wave equation.

$$f_x = g'(x + ct) \text{ and } f_{xx} = g''(x + ct). \text{ Meanwhile, } f_t = cg'(x + ct) \text{ and } f_{tt} = c^2 g''(x + ct).$$