Math 2130 Workshop: Projecting Circles
In this workshop, we’ll be practicing our three dimensional geometry skills to work out what the shadow of a circle looks like at an angle. Imagine the following setup:

- A light source $O$ at the origin $(0,0,0)$.
- A tilted circle $C$ given by all points of the form $(2 + \cos(\theta), \cos(\theta), \sqrt{2}\sin(\theta))$ for $0 \leq \theta \leq 2\pi$.
- A sheet of paper $P$ given by $x = 3$ onto which we will project the circle.

1) Verify that the circle $C$ has radius $\sqrt{2}$ by computing the distance between an arbitrary point on the circle $(2 + \cos(\theta), \cos(\theta), \sqrt{2}\sin(\theta))$ and the center $(2, 0, 0)$.

The distance is $\sqrt{(2 + \cos \theta - 2)^2 + \cos \theta^2 + (\sqrt{2}\sin \theta)^2} = \sqrt{2\cos^2 \theta + 2\sin^2 \theta} = \sqrt{2}$.

2) Verify that the circle $C$ lies in the plane $x - y = 2$, by plugging the $x$ and $y$ coordinates of an arbitrary point on the circle into the equation of the plane.

$2 + \cos \theta - \cos \theta$ is in fact 2.

3) Solve for the point where the circle $C$ intersects the plane $P$ by plugging in the coordinates of an arbitrary point on the circle into the equation of $P$ and solving.

We’re solving when $x = 2 + \cos \theta = 3$. This happens when $\cos \theta = 1$, which corresponds to the point $(3, 1, 0)$

4) Find an equation for a lightbeam $L$ passing through the light source at the origin $(0,0,0)$ and the point $(2,0,\sqrt{2})$, which is on the circle $C$.

In parameterized coordinates, we have $(2t, 0, \sqrt{2}t)$, for $-\infty < t < \infty$.

5) Find where the lightbeam $L$ hits the sheet of paper $P$ by solving for where they intersect.

We’re solving for when $\sqrt{2}t = 3$, which happens when $t = \frac{3}{\sqrt{2}}$, which happens at the point $(\frac{6}{\sqrt{2}}, 0, 3)$.
Now that we’re warmed up, let’s figure out what the shadow of the circle looks like.

6) Find an equation for a lightbeam \( L \) passing through the light source at the origin \((0, 0, 0)\) and an arbitrary point on the circle \((2 + \cos(\theta), \cos(\theta), \sqrt{2} \sin(\theta))\). Your answer will look something like: \((\Box t + \Box, \Box t + \Box, \Box t + \Box)\), where the expressions in the boxes may depend on \(\theta\).

\[
(t(2 + \cos \theta), t(\cos \theta), t\sqrt{2} \sin \theta). \text{ For } t \geq 0.
\]

7) Solve for the coordinates of the point where the lightbeam above intersects the sheet of paper \( P \) given by \( x = 3 \). Your answer will depend on \(\theta\).

We’re trying to find a \( t \) such that \( t(2 + \cos \theta) = 3 \). This happens when \( t = \frac{3}{2 + \cos \theta} \).

Plugging back in we get:

\[
\left(3, \frac{3 \cos \theta}{2 + \cos \theta}, \frac{3\sqrt{2} \sin \theta}{2 + \cos \theta}\right).
\]

8) Verify that the points given above satisfy the equation \( \frac{1}{4}(y + 1)^2 + \frac{1}{6}z^2 = 1 \), thus showing that the shadow of our circle is an ellipse.

Plugging in:

\[
\frac{1}{4} \left(2 + 4 \cos \theta\right)^2 + \frac{1}{6} \left(3\sqrt{2} \sin \theta\right)^2
= \frac{(1 + 2 \cos \theta)^2}{(2 + \cos \theta)^2} + \frac{3 \sin^2 \theta}{(2 + \cos \theta)^2}
= \frac{1 + 4 \cos \theta + 4 \cos^2 \theta}{4 + 4 \cos \theta + \cos^2 \theta} + \frac{3 \sin^2 \theta}{4 + 4 \cos \theta + \cos^2 \theta}
= \frac{1 + 4 \cos \theta + 4 \cos^2 \theta + 3 \sin^2 \theta}{4 + 4 \cos \theta + \cos^2 \theta}
= 1
\]
As we can see in this picture (the shadow of a square grid passing through the center of the circle is also included, for reference), it isn’t obvious that the shadow of the circle (a projected circle) will wind up being an ellipse (a stretched circle). A similar technique to the one in this exercise is used in computer graphics.