Math 2130 Workshop: Reasoning about Regular Tetrahedra II

Don’t forget: you don’t need to finish this assignment to get full credit. Just get through as much of this assignment as you can.

1) We will begin this workshop with a very abstract consideration of tetrahedra. Suppose we have a regular tetrahedron whose vertices are at the four points $\vec{0}, \vec{a}, \vec{b}, \vec{c}$. For each of the six edges, write out a vector corresponding to it.

$$\vec{a}, \vec{b}, \vec{c}, \vec{a} - \vec{b}, \vec{b} - \vec{c}, \text{ and } \vec{a} - \vec{c}.$$ 

2) Suppose now that the lengths of all of the edges are 1. Recalling that $||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}$, write out equations that must hold involving the dot product. Fill out the following multiplication table:

Since $||\vec{a}|| = 1$, we have that $\vec{a} \cdot \vec{a} = 1$, similar for the other two vectors. Since $||\vec{a} - \vec{b}|| = 1$, we have that $(\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = 1$. Expanding out the left hand side, we get:

$$\vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} = 1.$$ 

Rewriting this with the dot products we know, we get:

$$2 - 2\vec{a} \cdot \vec{b} = 1.$$ 

Or, $\vec{a} \cdot \vec{b} = \frac{1}{2}$. Every dot product of different vectors is the same for the same reason.

<table>
<thead>
<tr>
<th>.</th>
<th>$\vec{a}$</th>
<th>$\vec{b}$</th>
<th>$\vec{c}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vec{a}$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\vec{b}$</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\vec{c}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
</tr>
</tbody>
</table>

3) Use your answer to problem 2 to work out the angle between two edges. Hopefully, you should get $\frac{\pi}{3}$, since the faces of a tetrahedron are equilateral triangles.

Recall that the angle $\theta$ between two vectors $\vec{a}, \vec{b}$ satisfies: $\cos \theta = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}|| ||\vec{b}||}$. From this we can conclude that $\cos \theta = \frac{1}{2}$, or $\theta = \frac{\pi}{3}$. 

4) To calculate the angle between two faces, calculate the angle between the vector from \( \vec{b} \) to \( \vec{a} \) and the vector from \( \vec{b} \) to \( \vec{c} \). Use the properties of dot products and your table from problem 2 to work out this angle.

We’re trying to find the angle \( \theta \) between \( (\vec{c} - \frac{\vec{b}}{2}) \) and \( (\vec{a} - \frac{\vec{b}}{2}) \). Using the dot product formula, we have that:

\[
\cos \theta = \frac{(\vec{c} - \frac{\vec{b}}{2}) \cdot (\vec{a} - \frac{\vec{b}}{2})}{||\vec{c} - \frac{\vec{b}}{2}|| ||\vec{a} - \frac{\vec{b}}{2}||}
\]

The length of \( \vec{c} - \frac{\vec{b}}{2} \) is \( \sqrt{(\vec{c} - \frac{\vec{b}}{2}) \cdot (\vec{c} - \frac{\vec{b}}{2})} = \sqrt{\vec{c} \cdot \vec{c} - \vec{c} \cdot \vec{b} + \frac{\vec{b} \cdot \vec{b}}{4}} = \sqrt{1 - \frac{1}{2} + \frac{1}{4}} = \sqrt{\frac{3}{4}}. \)

The length of \( \vec{a} - \frac{\vec{b}}{2} \) is the same. The dot product in the numerator expands out to:

\[
(\vec{c} - \frac{\vec{b}}{2}) \cdot (\vec{a} - \frac{\vec{b}}{2}) = \vec{c} \cdot \vec{a} - \frac{1}{2} \vec{a} \cdot \vec{b} - \frac{1}{2} \vec{c} \cdot \vec{b} + \frac{1}{4} \vec{b} \cdot \vec{b} = \frac{1}{2} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4} = \frac{1}{4}.
\]

This gives us:

\[
\cos \theta = \frac{(\vec{c} - \frac{\vec{b}}{2}) \cdot (\vec{a} - \frac{\vec{b}}{2})}{||\vec{c} - \frac{\vec{b}}{2}|| ||\vec{a} - \frac{\vec{b}}{2}||} = \frac{\frac{1}{4}}{\sqrt{\frac{3}{4} \sqrt{\frac{3}{4}}} = \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}.
\]
5) In order to compute the volume of our tetrahedron, we need to use the formula
\[ \text{Volume} = \frac{1}{3} \text{Base} \cdot \text{Height}. \] The area of the base is \( \frac{1}{2} ||\vec{a} \times \vec{b}|| \). Why?

The base is half of the parallelogram formed by \( \vec{a} \) and \( \vec{b} \).

6) Compute the height of the tetrahedron by projecting \( \vec{c} \) onto a vector perpendicular to the base. Write out the formula for the volume in terms of \( \vec{a}, \vec{b} \) and \( \vec{c} \).

We want \( ||\vec{c}|| \cos \theta \) where \( \theta \) is the angle between \( \vec{c} \) and \( \vec{a} \times \vec{b} \). In other words, the height we’re looking for is \( \frac{\vec{c} \cdot (\vec{a} \times \vec{b})}{||\vec{a} \times \vec{b}||} \). The volume is \( \frac{1}{3} \) the base times height, or:

\[
\frac{1}{3} \left( \frac{1}{2} ||\vec{a} \times \vec{b}|| \right) \frac{\vec{c} \cdot (\vec{a} \times \vec{b})}{||\vec{a} \times \vec{b}||} = \frac{1}{6} (\vec{c} \cdot (\vec{a} \times \vec{b})).
\]

7) The points \((0,0,0), (1,1,0), (1,0,1), \) and \((0,1,1)\) form a regular tetrahedron. What is its edge-length? What is its volume?

Each of the four points is distance \( \sqrt{2} \) from each other.

\[
(\vec{i} + \vec{j}) \times (\vec{i} + \vec{k}) = \vec{i} \times \vec{j} + \vec{i} \times \vec{k} + \vec{j} \times \vec{i} + \vec{j} \times \vec{k} = 0 - \vec{j} - \vec{k} + \vec{i}.
\]

Dot producing this vector, \((1,−1,−1)\) with \((0,1,1)\) we get \(-2\), so the overall volume is \(\frac{1}{6} \cdot 2 = \frac{1}{3}\).

8) Find an \( x \) such that the points \((1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\) and \((x,x,x,x)\) are all equidistant. These points form a four-dimensional shape called a regular penta-choron.

The distance between \((1,0,0,0)\) and \((0,1,0,0)\) is \(\sqrt{2}\). So, we’re trying to solve the equation:

\[
\sqrt{(1 - x)^2 + x^2 + x^2 + x^2} = \sqrt{2},
\]

Or:

\[
(1 - x)^2 + x^2 + x^2 + x^2 = 2
\]

Or:

\[
4x^2 - 2x - 1 = 0
\]

Which has roots \( x = \frac{2 \pm \sqrt{4 + 16}}{8} = \frac{1 \pm \sqrt{5}}{4} \).