**Math 2130 Workshop: Reasoning about Regular Tetrahedra II**

Don't forget: you don't need to finish this assignment to get full credit. Just get through as much of this assignment as you can.

**1)** We will begin this workshop with a very abstract consideration of tetrahedra. Suppose we have a regular tetrahedron whose vertices are at the four points $\vec{0}, \vec{a}, \vec{b}, \vec{c}$. For each of the six edges, write out a vector corresponding to it.

> $\vec{a}, \vec{b}, \vec{c}, \vec{a} - \vec{b}, \vec{b} - \vec{c}$, and $\vec{a} - \vec{c}$.

**2)** Suppose now that the lengths of all of the edges are 1. Recalling that $||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}$, write out equations that must hold involving the dot product. Fill out the following multiplication table:

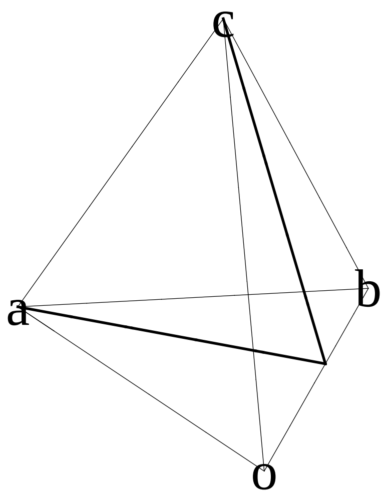
> Since $||\vec{a}|| = 1$, we have that $\vec{a} \cdot \vec{a} = 1$, similar for the other two vectors. Since $||\vec{a} - \vec{b}|| = 1$, we have that $(\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = 1$. Expanding out the left hand side, we get:
>
> $$\vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} = 1.$$
>
> Rewriting this with the dot products we know, we get:
>
> $$2 - 2\vec{a} \cdot \vec{b} = 1.$$
>
> Or, $\vec{a} \cdot \vec{b} = \frac{1}{2}$. Every dot product of different vectors is the same for the same reason.

| $\cdot$ | $\vec{a}$ | $\vec{b}$ | $\vec{c}$ |
|---|---|---|---|
| $\vec{a}$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $\vec{b}$ | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ |
| $\vec{c}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 |

**3)** Use your answer to problem **2** to work out the angle between two edges. Hopefully, you should get $\frac{\pi}{3}$, since the faces of a tetrahedron are equilateral triangles.

> Recall that the angle $\theta$ between two vectors $\vec{a}, \vec{b}$ satisfies: $\cos\theta = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||||\vec{b}||}$. From this we can conclude that $\cos\theta = \frac{1}{2}$, or $\theta = \frac{\pi}{3}$.

**4)** To calculate the angle between two faces, calculate the angle between the vector from $\frac{\vec{b}}{2}$ to $\vec{a}$ and the vector from $\frac{\vec{b}}{2}$ to $\vec{c}$. Use the properties of dot products and your table from problem **2** to work out this angle.




We're trying to find the angle $\theta$ between $(\vec{c} - \frac{\vec{b}}{2})$ and $(\vec{a} - \frac{\vec{b}}{2})$. Using the dot product formula, we have that:

$$\cos\theta = \frac{(\vec{c} - \frac{\vec{b}}{2}) \cdot (\vec{a} - \frac{\vec{b}}{2})}{||\vec{c} - \frac{\vec{b}}{2}||||\vec{a} - \frac{\vec{b}}{2}||}$$

The length of $\vec{c} - \frac{\vec{b}}{2}$ is $\sqrt{(\vec{c} - \frac{\vec{b}}{2}) \cdot (\vec{c} - \frac{\vec{b}}{2})} = \sqrt{\vec{c} \cdot \vec{c} - \vec{c} \cdot \vec{b} + \frac{\vec{b} \cdot \vec{b}}{4}} = \sqrt{1 - \frac{1}{2} + \frac{1}{4}} = \sqrt{\frac{3}{4}}$. The length of $\vec{a} - \frac{\vec{b}}{2}$ is the same. The dot product in the numerator expands out to:

$$(\vec{c} - \frac{\vec{b}}{2}) \cdot (\vec{a} - \frac{\vec{b}}{2}) = \vec{c} \cdot \vec{a} - \frac{1}{2}\vec{a} \cdot \vec{b} - \frac{1}{2}\vec{c} \cdot \vec{b} + \frac{1}{4}\vec{b} \cdot \vec{b} = \frac{1}{2} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4} = \frac{1}{4}.$$

This gives us:

$$\cos\theta = \frac{(\vec{c} - \frac{\vec{b}}{2}) \cdot (\vec{a} - \frac{\vec{b}}{2})}{||\vec{c} - \frac{\vec{b}}{2}||||\vec{a} - \frac{\vec{b}}{2}||} = \frac{\frac{1}{4}}{\sqrt{\frac{3}{4}}\sqrt{\frac{3}{4}}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

**5)** In order to compute the volume of our tetrahedron, we need to use the formula Volume $= \frac{1}{3}$Base $\cdot$ Height. The area of the base is $\frac{1}{2}||\vec{a} \times \vec{b}||$. Why?

The base is half of the parallelogram formed by $\vec{a}$ and $\vec{b}$.

**6)** Compute the height of the tetrahedron by projecting $\vec{c}$ onto a vector perpendicular to the base. Write out the formula for the volume in terms of $\vec{a}, \vec{b}$ and $\vec{c}$.

We want $||\vec{c}|| \cos\theta$ where $\theta$ is the angle between $\vec{c}$ and $\vec{a} \times \vec{b}$. In other words, the height we're looking for is $\frac{\vec{c}\cdot(\vec{a}\times\vec{b})}{||\vec{a}\times\vec{b}||}$. The volume is $\frac{1}{3}$ the base times height, or:

$$\frac{1}{3}(\frac{1}{2}||\vec{a} \times \vec{b}||\frac{\vec{c} \cdot (\vec{a} \times \vec{b})}{||\vec{a} \times \vec{b}||}) = \frac{1}{6}(\vec{c} \cdot (\vec{a} \times \vec{b})).$$

**7)** The points $(0, 0, 0), (1, 1, 0), (1, 0, 1)$, and $(0, 1, 1)$ form a regular tetrahedron. What is its edge-length? What is its volume?

Each of the four points is distance $\sqrt{2}$ from each other.

$$(\vec{i} + \vec{j}) \times (\vec{i} + \vec{k}) = \vec{i} \times \vec{i} + \vec{i} \times \vec{k} + \vec{j} \times \vec{i} + \vec{j} \times \vec{k} = 0 - \vec{j} - \vec{k} + \vec{i}.$$

Dot producting this vector, $(1, -1, -1)$ with $(0, 1, 1)$ we get $-2$, so the overall volume is $\frac{1}{6} \cdot 2 = \frac{1}{3}$.

**8)** Find an $x$ such that the points $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)$ and $(x, x, x, x)$ are all equidistant. These points form a four-dimensional shape called a *regular pentachoron*.

The distance between $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$ is $\sqrt{2}$. So, we're trying to solve the equation:

$$\sqrt{(1 - x)^2 + x^2 + x^2 + x^2} = \sqrt{2},$$

Or:

$$(1 - x)^2 + x^2 + x^2 + x^2 = 2$$

Or:

$$4x^2 - 2x - 1 = 0$$

Which has roots $x = \frac{2\pm\sqrt{4+16}}{8} = \frac{1\pm\sqrt{5}}{4}$.