Math 2130 Homework 9: 17.1-17.4

(1) Find the arclength of the curve $C$ parameterized by \( \vec{r}(t) = (t^2, \frac{4t^3}{3}, t) \) over the interval $0 \leq t \leq 1$.

\[
\vec{r}'(t) = (2t, 2\sqrt{t}, 1) \\
||\vec{r}'(t)|| = \sqrt{4t^2 + 4t + 1} \\
= \sqrt{(2t + 1)^2} \\
= |2t + 1| \\
= 2t + 1
\]

Where the last equality holds because $2t + 1 > 0$ on the interval $0 \leq t \leq 1$.

The integral we’re trying to compute is:

\[
\int_{t=0}^{t=1} 2t + 1 \ dt = [t^2 + t]_{t=0}^{t=1} \\
= 2
\]

(2) Find the average $y$-coordinate of the curve $C$ parameterized by $(\cos^3 t, \sin^3 t)$ over the interval $0 \leq t \leq \frac{\pi}{2}$.

The value we’re trying to compute is:

\[
\frac{\int_C y \, ds}{\int_C ds}
\]

We need to compute these separately, but first, let’s compute $||\vec{r}'(t)||$:

\[
\vec{r}'(t) = (-3\cos^2 t \sin t, 3\sin^2 t \cos t) \\
||\vec{r}'(t)|| = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} \\
= 3\sqrt{(\cos^2 t + \sin^2 t) \cos^2 t \sin^2 t} \\
= 3\sqrt{\cos^2 t \sin^2 t} \\
= 3|\cos t \sin t| \\
= 3\cos t \sin t
\]
Where the final equality holds because \( \cos t \sin t > 0 \) on the interval \( 0 \leq t \leq \frac{\pi}{2} \).

\[
\int_C y \, ds = \int_{t=0}^{t=\frac{\pi}{2}} \sin^3 t (3 \cos t \sin t) \, dt \\
= 3 \int_{t=0}^{t=\frac{\pi}{2}} \sin^4 t \cos t \, dt \\
= 3 \left[ \frac{\sin^5 t}{5} \right]_{t=0}^{t=\frac{\pi}{2}} \\
= \frac{3}{5}
\]

\[
\int_C ds = \int_{t=0}^{t=\frac{\pi}{2}} (3 \cos t \sin t) \, dt \\
= \left[ \frac{3}{2} \sin^2 t \right]_{t=0}^{t=\frac{\pi}{2}} \\
= \frac{3}{2}
\]

So the average value is \( \frac{3}{5} \cdot \frac{3}{2} = \frac{9}{10} \).

(3) Draw the vector field \( \vec{f}(x, y) = (y, x) \). If you want, print out the grids on the next page and submit them with your homework. You may also want to scale down your vectors to make the diagram nicer. If you do, be sure to scale down each vector by roughly the same amount.

(4) Draw the vector field \( \vec{f}(x, y) = (0, x^2) \).
(5) Show that the curve $\vec{r}(t) = (e^t - e^{-t}, e^t + e^{-t})$ is a flow curve for the vector field $\vec{f}(x, y) = (y, x)$.

\[ \vec{f}(\vec{r}(t)) = (e^t + e^{-t}, e^t - e^{-t}) = \vec{r}'(t) \]

(6) Describe qualitatively what happens if you try to approximate the flow lines of $f(x, y) = (-y, x)$ numerically. Specifically, how do the approximate flow lines relate to the actual flow lines? It will help to draw a picture.

As we can see in the above diagram, each step we move slightly further away from the center, so the approximation spirals away from the origin, while the actual flow lines are circular.
$\vec{f}(x, y) = (y, x)$:

$\vec{f}(x, y) = (0, x^2)$: