Math 2130 Homework 6: 15.1-15.3

(1) Find the absolute maxima and minima of $f(x, y) = x^2 + 2y^2 - x$ over the region $x^2 + y^2 \leq 1$. Use Lagrange multipliers on the boundary. Hint: you should have 5 potential candidate points.

We first look for critical points inside the region. $\nabla f = (2x - 1, 4y)$. This is $\vec{0}$ when $(x, y) = (\frac{1}{2}, 0)$.

Next, we look at the boundary. Using Lagrange Multipliers, we're looking for when $\nabla f = \lambda \nabla g$ where $g(x, y) = x^2 + y^2$. $\nabla g = (2x, 2y)$, so we're trying to solve the system of equations:

$$\begin{aligned} 2x - 1 =& \lambda 2x \\ 4y =& \lambda 2y \\ x^2 + y^2 =& 1 \end{aligned}$$

From the second equality, either $\lambda = 2$ or $y = 0$. If $y = 0$, then $x = \pm 1$, giving us the points $(1, 0)$ and $(-1, 0)$.

If $\lambda = 2$, then $2x - 1 = 4x$ so $x = -\frac{1}{2}$, and thus $y = \pm\frac{\sqrt{3}}{2}$, giving us the points $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$.

A table of points of interest and the value of f at those points is given below:

x	y	$f(x, y)$
$\frac{1}{2}$	0	$-\frac{1}{4}$
1	0	0
-1	0	2
$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{9}{4}$
$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{9}{4}$

So the absolute maximum is $\frac{9}{4}$, attained at $(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$ and the absolute minimum is $-\frac{1}{4}$ at $(-\frac{1}{2}, 0)$.

(2) For what values of k is $(0, 0)$ a local minimum of $f(x, y) = x^2 + kxy + y^2$? For what values is it a local maximum? For what values is it a saddle point?

$\nabla f = (2x + ky, 2y + kx)$ which is $\vec{0}$ at $(0, 0)$ so $(0, 0)$ is always one of a local maximum, a local minimum, or a saddle point. $f_{xx} = 2$, $f_{xy} = k$ and $f_{yy} = 2$, so $D = 4 - k^2$.

This is negative for $-2 < k < 2$, so, since $f_{xx}(0, 0) > 0$, $(0, 0)$ is a local minimum. If $k > 2$ or $k < -2$, then $D > 0$, so it is a saddle point.

If $k = 2$, then $f(x, y) = x^2 + 2xy + y^2 = (x + y)^2$, which is always ≥ 0. Since $f(0, 0) = 0$, it is a local minimum.

Similarly, if $k = -2$, then $f(x, y) = x^2 - 2xy + y^2 = (x - y)^2$, which is always ≥ 0. So similarly, $(0, 0)$ is a local minimum.

Summarizing, $(0, 0)$ is a local minimum for $-2 \leq k \leq 2$ and a saddle point for $k < -2$ and $k > 2$.

(3) Find the critical points of $f(x, y) = x^3 - y^3 - 3xy$, and classify them as local maxima, local minima, or saddle points.

$$f_x = 3x^2 - 3y$$
$$f_y = -3y^2 - 3x$$

These are 0 when $y = x^2$ and $y^2 = -x$, or $x^4 = -x$. This happens when $x^3 = -1$ (which happens when $x = -1$) or when $x = 0$. So our critical points are $(-1, 1)$ and $(0, 0)$.

$$f_{xx} = 6x$$
$$f_{yy} = -6y$$
$$f_{xy} = -3$$

At $(-1, 1)$ these are $-6, -6$ and -3 respectively, for a discriminant $D = (-6)(-6) - (-3)^2 = 27$. This is larger than 0, so $(-1, 1)$ is either a local max or local min. Since $f_{xx}(-1, 1) = -6 < 0$, the point is a local maximum.

At $(0, 0)$ these are $0, 0$ and -3 respectively, for a discriminant $D = 0 * 0 - (-3)^2 = -9$. This is smaller than 0, so $(0, 0)$ is a saddle point.

(4) Use Lagrange multipliers to find the minimum value of $x^2 + y^2 + z^2 + w^2$ given the constraint $ax + by + cz + dw = 1$ for some constants a, b, c and d.

The equation $\nabla f = \lambda \nabla g$ expands out to:

$$2x = \lambda a$$
$$2y = \lambda b$$
$$2z = \lambda c$$
$$2w = \lambda d$$

Plugging back into the constraint, we get that:

$$\frac{\lambda}{2} a^2 + \frac{\lambda}{2} b^2 + \frac{\lambda}{2} c^2 + \frac{\lambda}{2} d^2 = 1.$$

Solving for λ, we get $\lambda = \frac{2}{a^2+b^2+c^2+d^2}$, so:

$$x = \frac{a}{a^2 + b^2 + c^2 + d^2}$$
$$y = \frac{b}{a^2 + b^2 + c^2 + d^2}$$
$$z = \frac{c}{a^2 + b^2 + c^2 + d^2}$$
$$w = \frac{d}{a^2 + b^2 + c^2 + d^2}$$

Plugging back into our original function, we get:

$$\frac{a^2}{(a^2+b^2+c^2+d^2)^2} + \frac{b^2}{(a^2+b^2+c^2+d^2)^2} + \frac{c^2}{(a^2+b^2+c^2+d^2)^2} + \frac{d^2}{(a^2+b^2+c^2+d^2)^2}$$
$$= \frac{1}{a^2+b^2+c^2+d^2}.$$

(5) Find the minimum of $f(x,y) = -2x + 5x^2 - 4xy + y^2$ by first fixing x and solving for the y which, for that fixed x, minimizes $f(x,y)$. Call this $y(x)$. Then find the minimum value of $f(x, y(x))$.

First, fix x and look at the derivative with respect to y:

$$f_y(x,y) = -4x + 2y.$$

This is negative when $y < 2x$, zero when $y = 2x$ and positive when $y > 2x$, so $y = 2x$ is a global minimum. Plugging back in:

$$f(x,2x) = -2x + 5x^2 - 8x^2 + 4x^2 = -2x + x^2$$

Taking the derivative with respect to x:

$$\frac{d}{dx} f(x,2x) = -2 + 2x$$

This is negative when $x < 1$, zero when $x = 1$ and positive when $x > 1$, so $x = 1$ is a global minimum. Thus the minimum is at $(1,2)$, where the value of f is -1.