(1) Find the absolute maxima and minima of \( f(x, y) = x^2 + 2y^2 - x \) over the region \( x^2 + y^2 \leq 1 \). Use Lagrange multipliers on the boundary. Hint: you should have 5 potential candidate points.

We first look for critical points inside the region. \( \nabla f = (2x - 1, 4y) \). This is \( \vec{0} \) when \( (x, y) = (\frac{1}{2}, 0) \).

Next, we look at the boundary. Using Lagrange Multipliers, we’re looking for when \( \nabla f = \lambda \nabla g \) where \( g(x, y) = x^2 + y^2 \). \( \nabla g = (2x, 2y) \), so we’re trying to solve the system of equations:

\[
2x - 1 = \lambda 2x \\
4y = \lambda 2y \\
x^2 + y^2 = 1 
\]

From the second equality, either \( \lambda = 2 \) or \( y = 0 \). If \( y = 0 \), then \( x = \pm 1 \), giving us the points \( (1, 0) \) and \( (-1, 0) \).

If \( \lambda = 2 \), then \( 2x - 1 = 4x \) so \( x = -\frac{1}{2} \), and thus \( y = \pm \frac{\sqrt{3}}{2} \), giving us the points \( (-\frac{1}{2}, \frac{\sqrt{3}}{2}) \) and \( (-\frac{1}{2}, -\frac{\sqrt{3}}{2}) \).

A table of points of interest and the value of \( f \) at those points is given below:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( f(x, y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>( -\frac{1}{4} )</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>-( \frac{1}{2} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>9/4</td>
</tr>
<tr>
<td>-( \frac{1}{2} )</td>
<td>-( \frac{\sqrt{3}}{2} )</td>
<td>9/4</td>
</tr>
</tbody>
</table>

So the absolute maximum is \( \frac{9}{4} \), attained at \( (-\frac{1}{2}, \pm \frac{\sqrt{3}}{2}) \) and the absolute minimum is \( -\frac{1}{4} \) at \( (-\frac{1}{2}, 0) \).

(2) For what values of \( k \) is \( (0, 0) \) a local minimum of \( f(x, y) = x^2 + kxy + y^2 \)? For what values is it a local maximum? For what values is it a saddle point?

\( \nabla f = (2x + ky, 2y + kx) \) which is \( \vec{0} \) at \( (0, 0) \) so \( (0, 0) \) is always one of a local maximum, a local minimum, or a saddle point. \( f_{xx} = 2, f_{xy} = k \) and \( f_{yy} = 2 \), so \( D = 4 - k^2 \).

This is negative for \( -2 < k < 2 \), so, since \( f_{xx}(0, 0) > 0 \), \( (0, 0) \) is a local minimum. If \( k > 2 \) or \( k < -2 \), then \( D > 0 \), so it is a saddle point.

If \( k = 2 \), then \( f(x, y) = x^2 + 2xy + y^2 = (x + y)^2 \), which is always \( \geq 0 \). Since \( f(0, 0) = 0 \), it is a local minimum.

Similarly, if \( k = -2 \), then \( f(x, y) = x^2 - 2xy + y^2 = (x - y)^2 \), which is always \( \geq 0 \). So similarly, \( (0, 0) \) is a local minimum.

Summarizing, \( (0, 0) \) is a local minimum for \( -2 \leq k \leq 2 \) and a saddle point for \( k < -2 \) and \( k > 2 \).
(3) Find the critical points of \( f(x, y) = x^3 - y^3 - 3xy \), and classify them as local maxima, local minima, or saddle points.

\[
\begin{align*}
  f_x &= 3x^2 - 3y \\
  f_y &= -3y^2 - 3x
\end{align*}
\]

These are 0 when \( y = x^2 \) and \( y^2 = -x \), or \( x^4 = -x \). This happens when \( x^3 = -1 \) (which happens when \( x = -1 \)) or when \( x = 0 \). So our critical points are \((-1, 1)\) and \((0, 0)\).

\[
\begin{align*}
  f_{xx} &= 6x \\
  f_{yy} &= -6y \\
  f_{xy} &= -3
\end{align*}
\]

At \((-1, 1)\) these are \(-6, -6\) and \(-3\) respectively, for a discriminant \( D = (-6)(-6) - (-3)^2 = 27 \). This is larger than 0, so \((-1, 1)\) is either a local max or local min. Since \( f_{xx}(-1, 1) = -6 < 0 \), the point is a local maximum.

At \((0, 0)\) these are 0, 0 and \(-3\) respectively, for a discriminant \( D = 0 \ast 0 - (-3)^2 = -9 \). This is smaller than 0, so \((0, 0)\) is a saddle point.

(4) Use Lagrange multipliers to find the minimum value of \( x^2 + y^2 + z^2 + w^2 \) given the constraint \( ax + by + cz + dw = 1 \) for some constants \( a, b, c \) and \( d \).

The equation \( \nabla f = \lambda \nabla g \) expands out to:

\[
\begin{align*}
  2x &= \lambda a \\
  2y &= \lambda b \\
  2z &= \lambda c \\
  2w &= \lambda d
\end{align*}
\]

Plugging back into the constraint, we get that:

\[
\frac{\lambda}{2}a^2 + \frac{\lambda}{2}b^2 + \frac{\lambda}{2}c^2 + \frac{\lambda}{2}d^2 = 1
\]

Solving for \( \lambda \), we get \( \lambda = \frac{2}{a^2 + b^2 + c^2 + d^2} \), so:

\[
\begin{align*}
  x &= \frac{a}{a^2 + b^2 + c^2 + d^2} \\
  y &= \frac{b}{a^2 + b^2 + c^2 + d^2} \\
  z &= \frac{c}{a^2 + b^2 + c^2 + d^2} \\
  w &= \frac{d}{a^2 + b^2 + c^2 + d^2}
\end{align*}
\]
Plugging back into our original function, we get:

$$\frac{a^2}{(a^2 + b^2 + c^2 + d^2)^2} + \frac{b^2}{(a^2 + b^2 + c^2 + d^2)^2} + \frac{c^2}{(a^2 + b^2 + c^2 + d^2)^2} + \frac{d^2}{(a^2 + b^2 + c^2 + d^2)^2} = \frac{1}{a^2 + b^2 + c^2 + d^2}.$$ 

(5) Find the minimum of \( f(x, y) = -2x + 5x^2 - 4xy + y^2 \) by first fixing \( x \) and solving for the \( y \) which, for that fixed \( x \), minimizes \( f(x, y) \). Call this \( y(x) \). Then find the minimum value of \( f(x, y(x)) \).

First, fix \( x \) and look at the derivative with respect to \( y \):

\[
f_y(x, y) = -4x + 2y.
\]

This is negative when \( y < 2x \), zero when \( y = 2x \) and positive when \( y > 2x \), so \( y = 2x \) is a global minimum. Plugging back in:

\[
f(x, 2x) = -2x + 5x^2 - 8x^2 + 4x^2 = -2x + x^2
\]

Taking the derivative with respect to \( x \):

\[
\frac{d}{dx}f(x, 2x) = -2 + 2x
\]

This is negative when \( x < 1 \), zero when \( x = 1 \) and positive when \( x > 1 \), so \( x = 1 \) is a global minimum. Thus the minimum is at \((1, 2)\), where the value of \( f \) is \(-1\).