Math 2130 Homework 5: 14.6-14.7

(1) A point located at distance $r$ from the origin with angle $\theta$ counter-clockwise from the $x$-axis has $x$ coordinate $r \cos \theta$. Suppose $r$ and $\theta$ are functions of $t$. Write out a chain rule for $\frac{dx}{dt}$. Suppose $r$ is increasing at a constant rate of one unit per second, and $\theta$ is increasing at a constant rate of one unit per second (spiralling away from the origin). Use your chain rule to write $\frac{dx}{dt}$ as a function of $r$ and $\theta$. (Normally you would write it in terms of $x$ and $y$.)

\[
\frac{dx}{dt} = \frac{\partial x}{\partial r} \frac{dr}{dt} + \frac{\partial x}{\partial \theta} \frac{d\theta}{dt}.
\]

We have that: $\frac{\partial x}{\partial r} = \cos \theta$ and $\frac{\partial x}{\partial \theta} = -r \sin \theta$, and $\frac{dr}{dt} = 1$ and $\frac{d\theta}{dt} = 1$. Therefore:

\[
\frac{dx}{dt} = \cos \theta - r \sin \theta.
\]

(2) The temperature in a room at position $x, y, z$ relative to a heat source is given by $H(x, y, z) = (x^2 + y^2 + z^2)^{-\frac{3}{2}}$. A fly is flying around in the helix $(\cos t, \sin t, t)$ for $-2\pi \leq t \leq 2\pi$. Work out, using the multivariable chain rule, the rate of change in temperature of the fly at $t = \pi$.

The relevant chain rule is:

\[
\frac{\partial H}{\partial t} = \frac{\partial H}{\partial x} \frac{dx}{dt} + \frac{\partial H}{\partial y} \frac{dy}{dt} + \frac{\partial H}{\partial z} \frac{dz}{dt}.
\]

We can work out that

\[
\frac{\partial H}{\partial x} = -x(x^2 + y^2 + z^2)^{-\frac{3}{2}},
\]

\[
\frac{\partial H}{\partial y} = -y(x^2 + y^2 + z^2)^{-\frac{3}{2}},
\]

\[
\frac{\partial H}{\partial z} = -z(x^2 + y^2 + z^2)^{-\frac{3}{2}},
\]

\[
\frac{dx}{dt} = -\sin t,
\]

\[
\frac{dy}{dt} = \cos t,
\]

\[
\frac{dz}{dt} = 1.
\]

At $t = \pi$, $(x, y, z) = (-1, 0, \pi)$. Plugging this in, we have that:
\[
\frac{\partial H}{\partial x}(-1, 0, \pi) = (1 + \pi^2)^{-\frac{3}{2}}
\]
\[
\frac{\partial H}{\partial y}(-1, 0, \pi) = 0
\]
\[
\frac{\partial H}{\partial z}(-1, 0, \pi) = -\pi(1 + \pi^2)^{-\frac{3}{2}}
\]
\[
\frac{dx}{dt}(\pi) = 0
\]
\[
\frac{dy}{dt}(\pi) = 1
\]
\[
\frac{dz}{dt}(\pi) = 1
\]

Plugging everything in to our original equation, we get that
\[
\frac{\partial H}{\partial t}|_{t=\pi} = -\pi(1 + \pi^2)^{-\frac{3}{2}}.
\]

(3) The temperature in a room at position \(x, y, z\) relative to a heat source is given by \(H(x, y, z) = (x^2 + y^2 + z^2)^{-\frac{1}{2}}\). A fly is flying around in the helix \((\cos t, \sin t, t)\) for \(-2\pi \leq t \leq 2\pi\). Work out, by plugging in the formulas for \(x(t), y(t),\) and \(z(t)\) into the equation for \(H\) to directly get \(H\) as a function of \(t\), the rate of change in temperature of the fly at \(t = \pi\).

Plugging in, we get:
\[
H(t) = (\cos^2 t + \sin^2 t + t^2)^{-\frac{1}{2}} = (1 + t^2)^{-\frac{1}{2}}.
\]

Using the single variable calculus chain rule, we get:
\[
\frac{dH}{dt} = -t(1 + t^2)^{-\frac{3}{2}}
\]

Which is \(-\pi(1 + \pi^2)^{-\frac{3}{2}}\), when we plug in \(t = \pi\).

(4) Verify that \(f_{xy} = f_{yx}\) for \(f(x, y) = x^y\)

\[
f_x = yx^{y-1}.\text{ Then taking the partial derivative with respect to } y, \text{ we get: }
\]
\[
f_{xy} = x^{y-1} + yx^{y-1} \ln x.
\]

Meanwhile, \(f_y = x^y \ln x\). Taking the partial derivative with respect to \(x\), we get:
\[
f_{yx} = yx^{y-1} \ln x + x^y \frac{1}{x}.
\]

As desired, these are equal.

(5) Use a quadratic approximation for \(f(x, y) = x^y\) around \((x, y) = (3, 2)\) to approximate 3.021.99.
\[ f(3, 2) = 9 \]
\[ f_x(x, y) = yx^{y-1} \]
\[ f_x(3, 2) = 6 \]
\[ f_y(x, y) = x^y \ln(x) \]
\[ f_y(3, 2) = 9 \ln(3) \]
\[ f_{xx}(x, y) = y(y - 1)x^{y-2} \]
\[ f_{xx}(3, 2) = 2 \]
\[ f_{xy}(x, y) = x^{y-1} + yx^{y-1} \ln x \]
\[ f_{xy}(3, 2) = 3 + 6 \ln 3 \]
\[ f_{yy}(x, y) = x^y (\ln(x))^2 \]
\[ f_{yy}(3, 2) = 9(\ln 3)^2 \]

As such, the equation of our quadratic approximation is:

\[
Q(x, y) = 9 + 6(x - 3) + 9 \ln(3)(y - 2) + (x - 3)^2 + (3 + 6 \ln 3)(x - 3)(y - 2) + \frac{9}{2}(\ln(3))^2(y - 2)^2.
\]

Plugging in (3.02, 1.99), we get:

\[
Q(3.02, 1.99) = 9 + 6(.02) + 9 \ln(3)(-.01) + (.02)^2 + (3 + 6 \ln 3)(.02)(-.01) + \frac{9}{2}(\ln(3))^2(-.01)^2
\]

For comparison, this is \(Q(3.02, 1.99) = 9.020149\ldots\). The exact value is 9.020151\ldots.