Math 2130 Homework 13: 19.1-19.4, 21.3

(1) The divergence of the vector field:

$$\vec{F}(x,y,z) = \left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right)$$

is given by:

$$\nabla \cdot \vec{F} = \begin{cases} 0 & (x,y,z) \neq (0,0,0), \\ \text{undefined} & (x,y,z) = (0,0,0). \end{cases}$$

Compute the flux of the vector field $\vec{F}$ out of the unit sphere. Use the formulas:

$$\int_S \vec{F} \cdot d\vec{A} = \int_S \vec{F} \cdot \vec{n}\, dS$$
$$\int_S 1 dS = \text{Area}(S)$$

Where $\vec{n}$ is the unit outward normal.

(2) Check your answer above using the standard parameterization for the unit sphere using the formula:

$$d\vec{A} = \left(\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi} \right) d\phi\, d\theta = \sin\phi(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)\, d\phi\, d\theta.$$

(3) Use the divergence theorem and your answer to the above problem to find the flux of the vector field $\vec{F}$ (as in the problems above) out of the surface of the cube $-2 \leq x \leq 2, -2 \leq y \leq 2, -2 \leq z \leq 2$.

(4) Set up, but do not evaluate, an integral to compute the flux of the vector field $\vec{F}$ out of the bottom face of the cube (with corners $(-2,-2,-2), (-2,2,-2), (2,2,-2)$, and $(2,-2,-2)$).

(5) A new vector field $\vec{F}(x,y,z)$ satisfies the following:
- The flux of $\vec{F}$ out of the unit sphere is π.
- The divergence of $\vec{F}$ is constant 1, except at the origin, where it is undefined.

Use the divergence theorem to find the flux of the new vector field $\vec{F}$ out of the surface of the cube $-2 \leq x \leq 2, -2 \leq y \leq 2, -2 \leq z \leq 2$.

(6) Verify the divergence theorem for $\vec{F}(x,y,z) = (xy, yz, z)$ for the cylinder $x^2+y^2 \leq 1$ and $0 \leq z \leq 1$. Hint: don't forget the top and bottom surfaces, which you can also use the surface of revolution parameterization for.

(7) What happens if you try to compute the flux of the vector field $\vec{F}(x,y,z) = (0,0,-1)$ through the (upward-oriented) surface of the graph $z = f(x,y)$ over the region $a \leq x \leq b$, $c \leq y \leq d$?

(8) A Möbius strip is parameterized by:

$$\vec{r}(s,t) = \left((1 + s\cos\frac{t}{2})\cos t, (1 + s\cos\frac{t}{2})\sin t, s\sin\frac{t}{2} \right)$$

For $-1 \leq s \leq 1$ and $0 \leq t \leq 2\pi$. A picture is shown below. Work out the cross products to find what happens to $\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}$ when $(s,t) = (0,0)$ and $(s,t) = (0,2\pi)$. Hint: it may help to plug in the values of s and t as early as possible (for the variable you're taking the partial derivative with respect to, you need to wait until after taking the derivative; for the other plug in as your first step).