Math 2130 Homework 12: 19.3, 21.1, Scalar Surface Integrals

(1) Compute the divergence of the vector field \( \vec{F}(x, y) = \left( \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right) \).

\[
(F_1)_x + (F_2)_y = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} + \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} = \begin{cases} 
0 & (x, y) \neq (0, 0) \\
\text{undefined} & (x, y) = (0, 0)
\end{cases}
\]

(2) Compute the flux of the vector field \( \vec{F}(x, y) = \left( \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right) \) out of the unit circle.

We can’t use the divergence theorem here, so let’s use a vector line integral:

\[
\int_C (-F_2, F_1) \cdot d\vec{r} = \int_{t=0}^{t=2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t) dt = \int_{t=0}^{t=2\pi} 1 dt = 2\pi.
\]

(3) Use the divergence theorem to find the flux of the vector field \( \vec{F}(x, y) = \left( \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right) \) out of the square with vertices \((-2, -2), (-2, 2), (2, 2), (2, -2)\).

Let \( C_1 \) be the unit circle, \( C_2 \) be the square (both oriented counterclockwise), and \( R \) be the region between them. By the divergence theorem for the region \( R \):

\[
\int_R \text{div}(\vec{F}) \, dA = \int_{C_2} \vec{F} \cdot d\vec{r} - \int_{C_1} \vec{F} \cdot d\vec{r}
\]

The left hand side is 0, so \( \int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} = 2\pi. \)

(4) Set up (but do not evaluate) the flux integral out of the bottom edge of the square above.

We parameterize the bottom edge by \( \vec{r}(t) = (t, -2) \) for \(-2 \leq t \leq 2\). The flux integral is:

\[
\int_{t=-2}^{t=2} \left( \frac{2}{t^2 + 4} - \frac{t}{t^2 + 4} \right) \cdot (1, 0) dt = \int_{t=-2}^{t=2} \frac{2}{t^2 + 4} dt
\]

Note that you can evaluate this integral fairly easily if you remember the derivative of arctan.

(5) Parameterize the plane passing through \((1, 0, 0), (0, 1, 0), \) and \((0, 0, 1)\). For what values of the parameters \(s, t\) is \( \vec{r}(s, t) \) in the region \( x \geq 0, y \geq 0, z \geq 0 \)?

There are many solutions for this (sorry Iian!), including:

\[
\vec{r}(s, t) = (1, 0, 0) + (-1, 1, 0)s + (-1, 0, 1)t = (1 - s - t, s, t)
\]

For \(-\infty < s < \infty \) and \(-\infty < t < \infty \). For \(s \geq 0, t \geq 0\) and \(s + t \leq 1\) (ie. \(0 \leq t \leq 1 - s\)) all three coordinates are nonnegative.

(6) Parameterize the surface of a sphere of radius 2 centered at \((1, 0, 0)\).
\[ \vec{r}(\phi, \theta) = (2 \sin \phi \cos \theta + 1, 2 \sin \phi \sin \theta, 2 \cos \phi) \]
For \(0 \leq \phi \leq \pi\) and \(0 \leq \theta \leq 2\pi\).

(7) Parameterize the paraboloid \(z = x^2 + y^2\) using the formula for a graph and the formula for a surface of revolution. What do the parameter curves of each of these parameterizations look like?

As a graph, we can parameterize it by \(\vec{r}(s, t) = (s, t, s^2 + t^2)\) for \(-\infty < s < \infty, -\infty < t < \infty\). As a surface of revolution we can parameterize it by \(\vec{r}(t, \theta) = (t \cos \theta, t \sin \theta, t^2)\) for \(0 < t < \infty\) and \(0 \leq \theta \leq 2\pi\). The parameter curves of the first form a grid of parabolas, while the parameter curves of the second are circular and radial, as in the picture below:

(8) Parameterize the donut given by revolving the circle of radius 1 centered at \((2, 0)\) around the \(z\)-axis.

We parameterize the circle in the \(xz\)-plane as \(x(t) = 2 + \cos t\) and \(z(t) = \sin t\) for \(0 \leq t \leq 2\pi\). Using the formula for surfaces of revolution, we get:
\[ \vec{r}(t, \theta) = ((2 + \cos t) \cos \theta, (2 + \cos t) \sin \theta, \sin t) \]
For \(0 \leq t \leq 2\pi\) and \(0 \leq \theta \leq 2\pi\).

(9) Set up and evaluate an integral to find the average \(z\) coordinate of the conical surface \(z = 2 - 2\sqrt{x^2 + y^2}\) for \(z \geq 0\).
We parameterize the cone as a surface of revolution: \( \vec{r}(t, \theta) = (t \cos \theta, t \sin \theta, 2 - 2t) \) for \( 0 \leq t \leq 1 \) and \( 0 \leq \theta \leq 2\pi \). Computing the partial derivatives we get:

\[
\frac{\partial \vec{r}}{\partial t} = (\cos \theta, \sin \theta, -2) \\
\frac{\partial \vec{r}}{\partial \theta} = (-t \sin \theta, t \cos \theta, 0)
\]

Which has cross product:

\[
\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial t} = \det \begin{pmatrix}
\vec{i} & \vec{j} & \vec{k} \\
\cos \theta & \sin \theta & -2 \\
-t \sin \theta & t \cos \theta & 0
\end{pmatrix} = (2t \cos \theta, 2t \sin \theta, t)
\]

\[
\left\| \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial t} \right\| = \sqrt{4t^2 \cos^2 \theta + 4t^2 \sin^2 \theta + t^2} = t \sqrt{\frac{5}{3}}
\]

The integrals we want to compute are:

\[
\int_S (2 - 2t) \, dA = \int_{\theta=0}^{\theta=2\pi} \int_{t=0}^{t=1} (2 - 2t)t \sqrt{5} \, dt \, d\theta = 2\pi \sqrt{\frac{5}{3}}
\]

\[
\int_S \, dA = \int_{\theta=0}^{\theta=2\pi} \int_{t=0}^{t=1} t \sqrt{5} \, dt \, d\theta = \sqrt{5} \pi
\]

And the average \( z \)-coordinate is:

\[
\frac{2\pi \sqrt{\frac{5}{3}}}{\sqrt{5} \pi} = \frac{2}{3}
\]