Math 2130: Scalar Surface Integrals

To our current collection of “Calc 3 style integrals”, we add the following:

**Definition.** Given a surface $S$ in 3D space and function $f$, we write the integral of $f$ over the surface $S$ as:

$$\int_S f \, dS.$$

We use this notation to refer to adding up the value of $f$ (times tiny pieces of surface area) across the surface $S$. For instance, if we want to find the surface area of $S$, we need to compute $\int_S 1 \, dS$. If $f$ gives us the mass density of $S$ at a point then $\int_S f \, dS$ is the total mass of the surface. If we want to find the average value of $x$ over a surface (the $x$-coordinate of the center of mass), we need to compute:

$$\frac{\int_S x \, dS}{\int_S dS}.$$

As usual, we need a tool to convert these “Calc 3 style integrals” into integrals we can actually compute:

**Definition.** If the surface $S$ is parameterized by $\vec{r}(s,t)$ for values of $s,t$ for $a \leq s \leq b$ and $c \leq t \leq d$, then:

$$\int_S f \, dS = \int_{t=c}^{t=d} \int_{s=a}^{s=b} f(\vec{r}(s,t)) \left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\| \, ds \, dt.$$

Note that, if $\vec{r}(s,t) = (r_1(s,t), r_2(s,t), r_3(s,t))$, then:

$$\frac{\partial \vec{r}}{\partial s} = \left( \frac{\partial r_1}{\partial s}, \frac{\partial r_2}{\partial s}, \frac{\partial r_3}{\partial s} \right).$$

Just as we have a multiplication factor of $||\vec{r}'(t)||$ outside the differentials of our scalar surface integrals, we have the term $\left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\|$ outside the differentials of our scalar surface integrals.

It’s useful to think that:

$$dS = \left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\| \, ds \, dt.$$

**Example.** We find the average $z$ coordinate over the upper half of the surface of the unit sphere:

The integral we’re trying to compute is:

$$\frac{\int_S z \, dS}{\int_S dS} = \frac{\int_S z \, dS}{2\pi}.$$

We begin by parameterizing the upper half of the unit sphere. This is given by plugging in $R = 1$ to our parameterization of the sphere of radius $R$, and modifying the range for $\phi$:

$$\vec{r}(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi), \quad 0 \leq \phi \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq 2\pi.$$

We now need to compute $\frac{\partial \vec{r}}{\partial \phi}$ and $\frac{\partial \vec{r}}{\partial \theta}$, and their cross product:
\[
\frac{\partial \vec{r}}{\partial \phi} = (\cos \phi \cos \theta, \cos \phi \sin \theta, -\sin \phi) \\
\frac{\partial \vec{r}}{\partial \theta} = (-\sin \phi \sin \theta, \sin \phi \cos \theta, 0)
\]

\[
\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta} = \det \begin{pmatrix}
\vec{i} & \vec{j} & \vec{k} \\
\cos \phi \cos \theta & \cos \phi \sin \theta & -\sin \phi \\
-\sin \phi \sin \theta & \sin \phi \cos \theta & 0
\end{pmatrix}
= 0\vec{i} + \sin^2 \phi \sin \theta \vec{j} + \cos \phi \sin \phi \cos^2 \theta \vec{k}
+ \sin^2 \phi \cos \theta \vec{i} + 0\vec{j} + \sin \phi \cos \phi \sin^2 \theta \vec{k}
= \sin^2 \phi \cos \theta \vec{i} + \sin^2 \phi \sin \phi \phi \vec{j} + \cos \phi \sin \phi \vec{k}
\]

\[
\left\| \frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta} \right\| = \sqrt{\sin^4 \phi \cos^2 \theta + \sin^4 \phi \sin^2 \theta + \cos^2 \phi \sin^2 \phi}
= \sqrt{\sin^4 \phi + \cos^2 \phi \sin^2 \phi}
= \sqrt{\sin^2 \phi}
= |\sin \phi|
= \sin \phi
\]

Where the last equality holds because \( \sin \phi > 0 \) on the interval \( 0 \leq \phi \leq \frac{\pi}{2} \). It’s useful to have computed this for general radius \( R \). The result is, not surprisingly, \( R^2 \sin \phi \).

Recall that we were in the middle of setting up an integral:

\[
\int_S z \, dS = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\frac{\pi}{2}} \cos \phi \sin \phi d\phi d\theta
\]

\[
= 2\pi \left[ \frac{1}{2} \sin^2 \phi \right]_{\phi=\frac{\pi}{2}}^{\phi=0}
= \pi
\]

Recall that actually we were in the middle of computing an average value:

\[
\frac{\int_S z \, dS}{\int_S dS} = \frac{\int_S z \, dS}{2\pi} = \frac{\pi}{2\pi} = \frac{1}{2}.
\]

**Example.** Compute the surface area of \( S \), the portion of the graph \( z = x^2 + y^2 \) above the disk \( R \), given by \( x^2 + y^2 \leq 1 \).

We’re trying to compute \( \int_S dS \). Being a graph, we can parameterize \( S \) by its \( x \) and \( y \) coordinates: \( \vec{r}(x, y) = (x, y, x^2 + y^2) \) (for \( x, y \) in the region \( R \)).

We now need to compute \( \frac{\partial \vec{r}}{\partial x} \) and \( \frac{\partial \vec{r}}{\partial y} \), and their cross product:
\[
\frac{\partial \vec{r}}{\partial x} = (1, 0, 2x) \\
\frac{\partial \vec{r}}{\partial y} = (0, 1, 2y)
\]

\[
\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = (\vec{i} + 2x\vec{k}) \times (\vec{j} + 2y\vec{k})
\]

\[
= \vec{i} \times \vec{j} + 2y\vec{i} \times \vec{k} + 2x\vec{k} \times \vec{j} + 4xy\vec{k} \times \vec{k}
\]

\[
= \vec{k} - 2y\vec{j} - 2x\vec{i}
\]

\[
= (-2x, -2y, 1)
\]

\[
\left\| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right\| = \sqrt{4x^2 + 4y^2 + 1}
\]

Now we can compute:

\[
\int_S dS = \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \left\| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right\| dy dx
\]

\[
= \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \sqrt{4x^2 + 4y^2 + 1} dy dx
\]

\[
= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} (\sqrt{4r^2 + 1}) r dr d\theta
\]

\[
= 2\pi \left[ \frac{1}{12} (1 + 4r^2)^{3/2} \right]_{r=0}^{r=1}
\]

\[
= \frac{\pi \sqrt{125}}{6} - \frac{\pi}{6}
\]
But why $\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \|$? You don’t need to know this, but it may be helpful for remembering the formula, and for reasoning about our next topic, flux integrals in 3D.

Let’s work out the surface area of the unit sphere. We already know the answer, so we can check if our method is correct. Recall that the surface of the unit sphere is parameterized by:

$$\vec{r}(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi).$$

**Step 1** in our approximation will be to locate a grid of latitude and longitude points on our sphere. These points will be given by $\vec{r}(n \Delta \phi, m \Delta \theta)$ for some small $\Delta \phi$ and $\Delta \theta$ and $n$ and $m$ integers. Shown below is the result for when we use 100 sample points and when we use 1600 sample points.

Step 2 in our approximation will be to approximate these small trapezoids with parallelograms. The corners of our trapezoids are given by:

$$\vec{r}(\phi, \theta), \vec{r}(\phi + \Delta \phi, \theta), \vec{r}(\phi + \Delta \phi, \theta + \Delta \theta), \vec{r}(\phi, \theta + \Delta \theta).$$

While we have a formula for the area of a trapezoid, we have a much nicer formula for the area of a parallelogram, and when applying this technique in general, it is not likely that nearby points will form trapezoids: in general they may not be coplanar! Instead of having the far corner of our shape be given by $\vec{r}(\phi + \Delta \phi, \theta + \Delta \theta)$, we will work out its location by looking at the other three corners and completing the parallelogram. Shown below is the result for 100 and 1600 sample points:

Notice that while it looks like our approximation is pretty bad for 100 sample points, the approximation looks much better for 1600 sample points, and keeps getting better as we increase the number of sample points.
We can work out the areas of these parallelograms using the cross product:

\[ \vec{r}(\phi, \theta) \quad \vec{r}(\phi, \theta + \Delta \theta) \]

\[ \vec{r}(\phi + \Delta \phi, \theta) \]

The area of the above parallelogram is the magnitude \( ||(\vec{r}(\phi, \theta + \Delta \theta) - \vec{r}(\phi, \theta)) \times (\vec{r}(\phi + \Delta \phi, \theta) - \vec{r}(\phi, \theta))|| \).

**Step 3** will be to approximate the edges of these parallelograms, using partial derivatives as “change magnification factors”.

\[
\vec{r}(\phi, \theta + \Delta \theta) - \vec{r}(\phi, \theta) \approx \frac{\partial \vec{r}}{\partial \theta} \Delta \theta
\]

\[
\vec{r}(\phi + \Delta \phi, \theta) - \vec{r}(\phi, \theta) \approx \frac{\partial \vec{r}}{\partial \phi} \Delta \phi
\]

Shown below are the pictures for 100 sample points and 1600 sample points with the edges replaced by their approximations above.

Notice again the left approximation looks really dire while adding more sample points fixes it. It’s easy to imagine that as the number of sample points approaches infinity, the sum of the areas of our approximate trapezoids approaches the actual surface area.

**Step 4** will be to add up the areas of our parallelograms. Note that by scalar multiplication rules for cross products and vector magnitudes:

\[
\left\| \left( \frac{\partial \vec{r}}{\partial \theta} \Delta \theta \right) \times \left( \frac{\partial \vec{r}}{\partial \phi} \Delta \phi \right) \right\| = \left\| \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi} \right\| \Delta \theta \Delta \phi
\]

So it makes sense that when we add these terms up and take the limit as \( \Delta \theta \) and \( \Delta \phi \) approach 0, we get:

\[
\int_{\phi=0}^{\phi=\pi} \int_{\theta=0}^{\theta=2\pi} \left\| \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi} \right\| d\theta d\phi
\]