Math 2130: Scalar Line Integrals
To our current collection of “Calc 3 style integrals”, we add the following:

**Definition.** Given a curve in space \(C\) and function \(f\), we write the integral of \(f\) over the curve \(C\) as:

\[
\int_C f \, ds.
\]

We use this notation to refer to adding up the value of \(f\) (times tiny pieces of arc length) along points of the curve \(C\). For instance, if we want to find the length of a curve, we need to compute \(\int_C 1 \, ds\). If \(f\) gives us the mass density of the curve \(C\) at a point, then \(\int_C f \, ds\) is the total mass of the curve. If we want to find the average value of \(x\) over a curve (the \(x\)-coordinate of the center of mass), we need to compute:

\[
\frac{\int_C x \, ds}{\int_C ds}.
\]

As before, we need a tool to convert these “Calc 3 style integrals” into integrals we can actually compute:

**Definition.** If the curve \(C\) is parameterized by \(\vec{r}(t)\) over the interval \(a \leq t \leq b\), then:

\[
\int_C f \, ds = \int_{t=a}^{t=b} f(\vec{r}(t)) \|\vec{r}'(t)\| \, dt.
\]

Just as we had a multiplication factor outside the differentials of our polar, cylindrical, and spherical integrals, we have a multiplication factor of \(\|\vec{r}'(t)\|\) outside the differential. It’s useful to think that:

\[
ds = \|\vec{r}'(t)\| \, dt
\]

**Example.** We compute the \(y\)-coordinate of the center of mass of the half-circle \(\vec{r}(t) = (\cos t, \sin t)\) for \(0 \leq t \leq \pi\).

We need to compute:

\[
\frac{\int_C y \, ds}{\int_C ds}.
\]

We recognize the denominator as the arc length of our curve, half the circumference of the circle, or \(\pi\). Note that \(\vec{r}'(t) = (-\sin t, \cos t)\) We compute the numerator as:

\[
\int_C y \, ds = \int_{t=0}^{t=\pi} y \sqrt{(-\sin t)^2 + (\cos t)^2} \, dt
\]

\[
= \int_{t=0}^{t=\pi} \sin t \, dt
\]

\[
= \left[ -\cos t \right]_{t=0}^{t=\pi}
\]

\[
= 2
\]

Going back to our original problem, we’re trying to compute \(\frac{\int_C y \, ds}{\int_C ds} = \frac{2}{\pi}\).

**Example.** We compute the \(y\)-coordinate of the center of mass of the half-circle \(\vec{r}(t) = (t, \sqrt{1-t^2})\) for \(-1 \leq t \leq 1\).
Note that this is the same problem as before, just with a different parameterization. Note in particular that this parameterization starts at \((-1, 0)\) and ends at \((1, 0)\) while the parameterization before started at \((1, 0)\) and ended at \((-1, 0)\).

In this case:

\[
\vec{r}'(t) = (1, \frac{-2t}{2\sqrt{1-t^2}}) = (1, \frac{-t}{\sqrt{1-t^2}})
\]

\[
||\vec{r}'(t)|| = \sqrt{1 + \frac{t^2}{1-t^2}} = \sqrt{\frac{1}{1-t^2}}
\]

We compute:

\[
\int_C y \, ds = \int_{t=-1}^{t=1} \sqrt{\frac{1}{1-t^2}} \sqrt{\frac{1}{1-t^2}} \, dt = \int_{t=-1}^{t=1} 1 \, dt = 2
\]

As before, we divide by the arc length \(\pi\) to get the final average \(y\)-coordinate of \(\frac{2}{\pi}\).

**Theorem.** The integral of a function over a curve does not depend on which parameterization you pick.

To get a sense of why this is true, let’s go over a very basic example:

**Example.** We add up the value of \(f(x)\) over the line segment from \((0, 0)\) to \((1, 0)\).

First we use the parameterization \(\vec{r}(t) = (t, 0)\) for \(0 \leq t \leq 1\): (note that \(\vec{r}'(t) = (1, 0)\) which has length 1.

\[
\int_C f \, ds = \int_{t=0}^{t=1} f(t) \left(1 \, dt\right)
\]

Now we add up the value of \(f(x)\) over the line segment from \((0, 0)\) to \((1, 0)\), using the parameterization \(\vec{r}(t) = (2t, 0)\) for \(0 \leq t \leq \frac{1}{2}\): (note that \(\vec{r}'(t) = (2, 0)\) which has length 2. We’re traversing the line segment twice as fast.)

\[
\int_C f \, ds = \int_{t=0}^{t=\frac{1}{2}} f(2t)2 \, dt
\]

So while the interval of values of \(t\) we’re integrating over is twice as short, our integrand is twice as large to compensate. Note that the \(u\)-substitution \(u = 2t\) gives us our first integral. In general, the larger \(||\vec{r}'(t)||\) is, the shorter the interval of values of \(t\) we need to integrate over.