Math 2130 Practice Final (Spring 2017)

Before the exam:

- Do not write anything on this page.
- Do not open the exam.
- Turn off your cell phone.
- Make sure your books, notes, and electronics are not visible during the exam.
- Do not wear headphones during the exam.

When you open your exam:

- Make sure your exam has all its pages. There are 8 pages, including the last, and 8 problems.
- If you believe there is a printing error, let me know right away.
- Write your name on the last page, and put a check in the box corresponding to your section.

During the exam:

- Do not talk or ask questions. If you are unsure what a question is asking, demonstrate your understanding as best you can.
- Be respectful of your fellow classmates.
- You may use the bathroom during the exam, but please ask first so I can keep track of who is out of the room at any one time.

Notes on grading:

- Draw a $\boxed{\text{box}}$ around your final solution to the problem.
- Show your work. Demonstrate that you know how to get the correct answer, not just make a lucky guess.
- Clearly cross out any work that is incorrect.
- Each problem is worth 10 points.
- If you run out of room, continue your work on the back of the previous page. Make a note that you've done this, and make it clear where your work continues.

Do not write in this box	
(1)	______
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(8)	______
Total	______

(1) Find the largest and smallest values of $f(x, y, z) = z^2 - xy$ over the solid (filled in) sphere of radius 2. Use any techniques we learned in this class. Show all your work. To make this problem easier to grade, be sure to include a complete table of points you considered in your search in one clearly labeled location.

We first check the interior of the sphere for critical points. $\nabla f = (-y, -x, 2z)$ which is $\vec{0}$ when $x = y = z = 0$.

We now check the boundary of the sphere for critical points using Lagrange Multipliers. Since the sphere is given by points where $g(x, y, z) = x^2+y^2+z^2 = 4$, we're looking for points where $\nabla f = \lambda \nabla g$, or:

$$\begin{aligned} -y =& 2\lambda x \\ -x =& 2\lambda y \\ 2z =& 2\lambda z \end{aligned}$$

The third equation has solution $z = 0$ or $\lambda = 1$.

Case 1: $z = 0$. Here, $x^2 + y^2 = 4$. Reversing the second equation and multiplying by the first, we get: $-2\lambda y^2 = -2\lambda x^2$. This splits into three cases, one where $\lambda = 0$, one where $x = y$, and the last when $x = -y$.

Case 1a: $z = 0$ and $\lambda = 0$. In this case, $(x, y, z) = (0, 0, 0)$. Not only is this point already in consideration, but it isn't even on the surface of the sphere.

Case 1b: $z = 0$ and $x = y$. In this case, we can solve to get the points $(\sqrt{2}, \sqrt{2}, 0)$ and $(-\sqrt{2}, -\sqrt{2}, 0)$.

Case 1c: $z = 0$ and $x = -y$. In this case, we can solve to get the points $(\sqrt{2}, -\sqrt{2}, 0)$ and $(-\sqrt{2}, \sqrt{2}, 0)$.

Case 2: $\lambda = 1$. In this case, the first two equations tell us $y = -2x$ and $x = -2y$. Plugging one equation into the other, we find the only solution is if $x = y = 0$, and thus $z = \pm 2$.

So overall we found the following points for consideration:

$$\begin{aligned} f(0,0,0) =& 0 \\ f(0,0,2) =& 4 \\ f(0,0,-2) =& 4 \\ f(\sqrt{2}, \sqrt{2}, 0) =& -2 \\ f(-\sqrt{2}, -\sqrt{2}, 0) =& -2 \\ f(-\sqrt{2}, \sqrt{2}, 0) =& 2 \\ f(\sqrt{2}, -\sqrt{2}, 0) =& 2 \end{aligned}$$

Thus the maximum value of f is 4 and the minimum value is -2.

(2) Find the largest and smallest values of $f(x, y, z) = 2z + x^2 + y^2$ over the triangular surface with corners $(4, 0, 0)$, $(0, 4, 0)$, and $(0, 0, 4)$. Use any techniques we learned in this class. Show all your work. To make this problem easier to grade, be sure to include a complete table of points you considered in your search in one clearly labeled location.

The triangular surface is given by $g(x, y, z) = x + y + z = 4$, so Lagrange multipliers gives us the equations:

$$\begin{aligned} 2x =& \lambda \\ 2y =& \lambda \\ 2 =& \lambda \end{aligned}$$

Which has solution $x = 1, y = 1, z = 2$.

However, we also have to check the corners and sides of the triangles.

Case 1: $z = 0, x + y = 4$. Plugging $y = 4 - x$ and $z = 0$ into our equation for $f(x, y, z)$, we get $2x^2 - 8x + 16$, which has a critical point when $4x - 8 = 0$ or $x = 2$, at which point $y = 2$ and $z = 0$.

Case 2: $y = 0, x + z = 4$. Plugging $z = 4 - x$ and $y = 0$ into our equation for $f(x, y, z)$, we get $x^2 - 2x + 8$, which has a critical point when $2x - 2 = 0$ or $x = 1$, at which point $z = 3$ and $y = 0$.

Case 3: $x = 0, y + z = 4$. This is just the above case with x and y switched, so we should expect the solution $x = 0, y = 1, z = 3$.

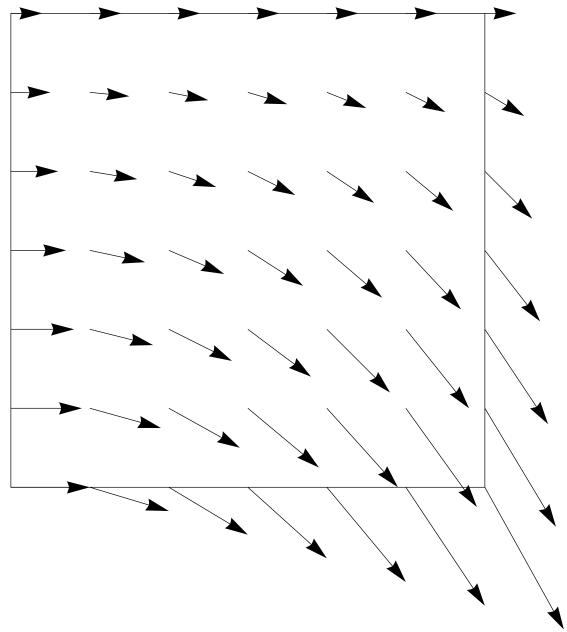
We additionally have to consider the corners of the triangle: $(4, 0, 0)$, $(0, 4, 0)$, and $(0, 0, 4)$.

So overall we found the following points for consideration:

$$\begin{aligned} f(1, 1, 2) =& 6 \\ f(2, 2, 0) =& 8 \\ f(1, 0, 3) =& 7 \\ f(0, 1, 3) =& 7 \\ f(4, 0, 0) =& 16 \\ f(0, 4, 0) =& 16 \\ f(0, 0, 4) =& 8 \end{aligned}$$

Thus the maximum value of f over this region is 16 and the minimum value is 6.

(3) Shown below is a portion of a vector field. For each of the sides of the square, does that side contribute a positive, negative, or zero term to the circulation integral counterclockwise around the square and the flux integral out of the square?



Side	Contribution to Circulation	Contribution to Flux Integral
Top	Negative	0
Left	0	Negative
Right	Negative	Positive
Bottom	Positive	Positive

(4) The vector field $\vec{F}$ satisfies:

- $\nabla \cdot \vec{F}$ is constant 3.
- The flux of $\vec{F}$ upwards through the unit disk in the xy plane is π.
- The flux of $\vec{F}$ outwards through the unit cylinder (the unit circle for $z = 0$ to $z = 1$) is 4.

What is the flux of $\vec{F}$ upwards through the unit disk in the plane $z = 1$?

The Divergence Theorem says that the integral of the divergence of $\vec{F}$ over the cylinder is equal to the net flux out of the surface of the cylinder. The integral of the divergence of $\vec{F}$ over the cylinder is just three times the volume of the cylinder, or 3π. The net flux of $\vec{F}$ out of the cylinder is equal to the flux downwards through the unit disk in the xy plane $(-\pi)$ plus the flux outwards through the unit cylinder (4) plus the flux of upwards through the unit disk in the plane $z = 1$. Letting this last flux be x, we have $3\pi = -\pi + 4 + x$ or $x = 4\pi - 4$.

(5) Verify Stokes' Theorem for the vector field $\vec{F} = (x + xy, y^2, yz)$ over the top half of the surface of the sphere of radius 2. Show every step of your work.

Let S be the surface of the sphere and C be its boundary, the circle of radius 2 in the xy plane. If we orient S upwards, we need to orient C counterclockwise to agree with the orientation of S. Stokes' Theorem says that

$$\int_S (\nabla \times \vec{F}) \cdot d\vec{A} = \int_C \vec{F} \cdot d\vec{r}.$$

Evaluating the curl of $\vec{F}$ we get $(z, 0, -x)$. Evaluating the left hand side with the parameterization $\vec{r}(\phi, \theta) = (2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi)$, with $0 \leq \phi \leq \frac{\pi}{2}$ and $0 \leq \theta \leq 2\pi$, we get:

$$\begin{aligned}\int_S (\nabla \times \vec{F}) \cdot d\vec{A} &= \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\frac{\pi}{2}} (2\cos\phi, 0, -2\sin\phi\cos\theta) \cdot (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) 4\sin\phi \, d\phi \, d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\frac{\pi}{2}} 0 \, d\phi \, d\theta \\ &= 0\end{aligned}$$

Evaluating the right hand side with parameterization $\vec{r}(t) = (2\cos t, 2\sin t, 0)$ for $0 \leq t \leq 2\pi$, we get:

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_{t=0}^{t=2\pi} (2\cos t + 4\cos t\sin t, 4\sin^2 t, 0) \cdot (-2\sin t, 2\cos t, 0) \, dt \\ &= \int_{t=0}^{t=2\pi} -4\cos t\sin t - 8\cos t\sin^2 t + 8\sin^2 t\cos t dt \\ &= \int_{t=0}^{t=2\pi} -4\cos t\sin t dt \\ &= \left[-2\sin^2 t\right]_{t=0}^{t=2\pi} \\ &= 0\end{aligned}$$

(6) Use a quadratic approximation to approximate $1.01^{1.98}$. You may leave any arithmetic operations unevaluated. Irrelevant to the problem, but of note, is that your approximation will be less than 3 millionths off from the actual value.

We're trying to approximate $f(1+.01, 2-.02)$ for $f(x,y) = x^y$. The partial derivatives are:

$$\begin{aligned}
f(1,2) =&(x^y)|_{(x,y)=(1,2)} = 1\\
f_x(1,2) =&(yx^{y-1})|_{(x,y)=(1,2)} = 2\\
f_y(1,2) =&(x^y \ln x)|_{(x,y)=(1,2)} = 0\\
f_{xx}(1,2) =&(y(y-1)x^{y-2})|_{(x,y)=(1,2)} = 2\\
f_{xy}(1,2) =&(x^{y-1} + yx^{y-1}\ln x)|_{(x,y)=(1,2)} = 1\\
f_{yy}(1,2) =&(x^y(\ln x)^2)|_{(x,y)=(1,2)} = 0
\end{aligned}$$

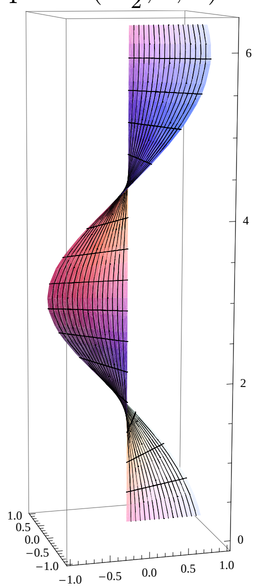
The formula for the quadratic approximation is:

$$\begin{aligned}
Q(x,y) =&f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) +\\
&+ f_{xx}(1,2)\frac{(x-1)^2}{2} + f_{xy}(1,2)(x-1)(y-2) + f_{yy}(1,2)\frac{(y-2)^2}{2}\\
=&1 + 2(x-1) + (x-1)^2 + (x-1)(y-2)\\
Q(1.01, 1.98) =&1 + 2(.01) + (.01)^2 + (.01)(-.02)\\
=&1 + .02 + .0001 - .0002\\
=&1.0199
\end{aligned}$$

(7) Is the vector field $\vec{F} = (yz, xz, xy)$ path independent?

Yes. It is the gradient of $f = xyz$.

(8) The helical surface shown below is parameterized by $\vec{r}(s,t) = (s\cos t, s\sin t, t)$ for $0 \leq s \leq 1$ and $0 \leq t \leq 2\pi$. Find the equation of a tangent plane to the surface at the point $(-\frac{1}{2}, 0, \pi)$.



First, note that the point $(-\frac{1}{2}, 0, \pi)$ happens when $s = \frac{1}{2}$ and $t = \pi$. We know how to compute a normal vector to the surface:

$$\begin{aligned}
\frac{d\vec{r}}{ds} \times \frac{d\vec{r}}{dt} =& (\cos t, \sin t, 0) \times (-s\sin t, s\cos t, 1) \\
=& \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos t & \sin t & 0 \\ -s\sin t & s\cos t & 1 \end{pmatrix} \\
=& (\sin t, -\cos t, s(\sin^2 t + \cos^2 t)) \\
=& (\sin t, -\cos t, s)
\end{aligned}$$

Plugging in the point, we get $(0, 1, \frac{1}{2})$.

If we have a point (x_0, y_0, z_0) and normal vector (a, b, c), the equation of the plane is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$. Plugging in our point and normal vector, we get:

$$y + \frac{1}{2}(z - \pi) = 0$$

Or, $z = -2y + \pi$

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Name: ______________________________ Netid: ________________

Section (check which one applies):

☐ Discussion 1 (9:05am-9:55am)

☐ Discussion 2 (10:10am-11:00am)