April 13, 2015

Prelim 2 Solutions V1 Math 2130

Please show your reasoning and all your work. This is a 50 minute exam. Calculators are not needed or permitted. Good luck!

Unless specifically requested, answers on this exam NEED NOT be simplified or completely numerical. Arithmetic expressions are fine. Ordinarily, answers like

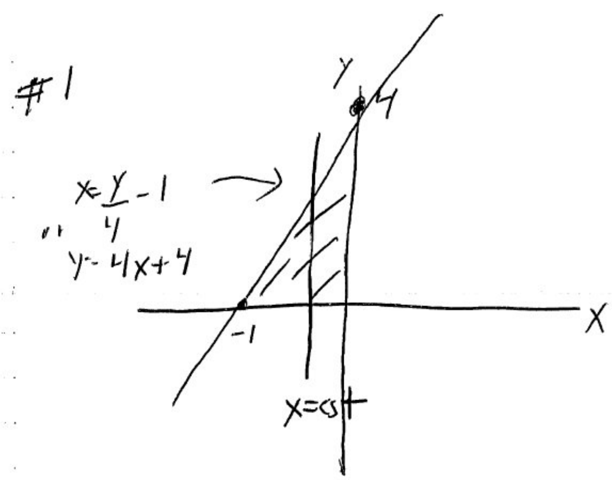
$$\ln\left(\frac{10+2}{5}\right), \qquad \frac{3x^2(x^4-1)-4x^3(x^3+1)}{(x^4-1)^2}, \qquad \frac{1}{2}\left(4x^3\right)\left(x^4+1\right)^{.5}$$

are completely acceptable.

Problem 1) (20 Points) Reverse the order of integration and use your new integral to find the value of

$$\int_0^4 \int_{\frac{y}{4}-1}^{0} e^{(x+1)^2}\, dx\, dy.$$

Solution: From the limits of integration, we get the picture



and setting up in the order $dy\ dx$ gives

$$\int_{-1}^{0} \int_{0}^{4x+4} e^{(x+1)^2}\ dy\ dx = \int_{-1}^{0} 4(x+1)e^{(x+1)^2}\ dx$$
$$= 2e^{(x+1)^2}\Big|_{-1}^{0} = 2(e-1).$$

Problem 2) (20 Points) For $\displaystyle\int_0^1 \int_x^{\sqrt{2-x^2}} dy\ dx$,

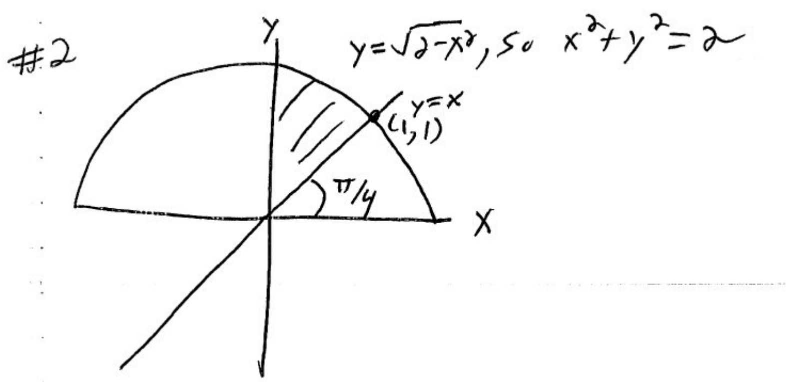
2a) Sketch the region in the xy plane being integrated over:

2b) Express the double integral as a double integral in polar coordinates.

2c) Use your expression in part 2b to find the value of the integral.

Solution:

2a)



2b) Using the above picture, we have

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\sqrt{2}} r\ dr\ d\theta.$$

2c)

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\sqrt{2}} r\ dr\ d\theta = \left(\frac{\pi}{2} - \frac{\pi}{4}\right)\left(\frac{r^2}{2}\Big|_0^{\sqrt{2}}\right) = \frac{\pi}{4}.$$

This problem resembles homework problem 22 in section 16.4.

Problem 3) (20 Points) Consider the (solid) region in R^3 between the planes $z = 8 - y$ and $z = 6 + y$ as well as above the triangle in the $xy-$plane with vertices $(0,0,0)$, $(4,0,0)$, and $(0,4,0)$. (We are viewing the positive z axis as the "up" direction, and by "between" we are also referring to vertical position.)

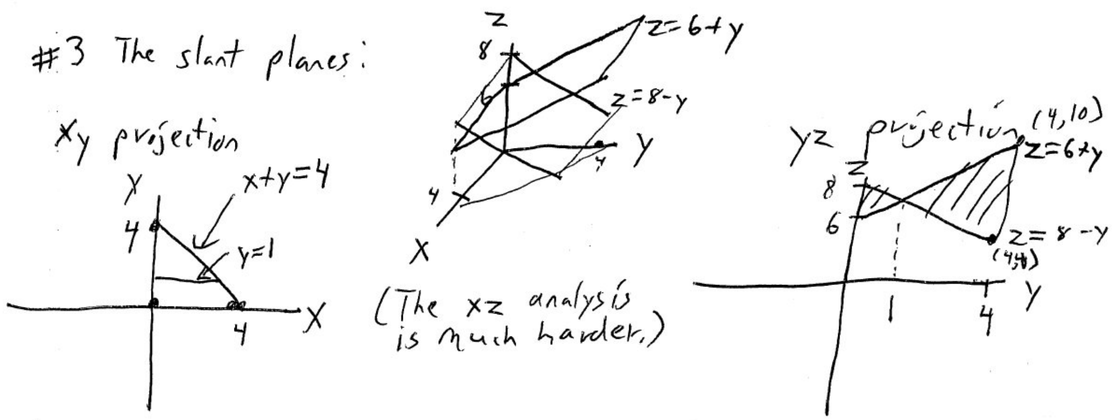
Let $f(x, y, z)$ be a differentiable function within this region (though not necessarily outside the region.) Set up as a sum of triple integrals the triple integral of $f(x, y, z)$ over this region. Include all limits of integration but do not attempt to compute the value of the integral.

Solution: The planes are easy to picture since algebraically they don't involve the variable x; consequently they are both lines in the yz plane "extruded" in the x-direction. Clearly those lines meet along the line where $8 - y = 6 + y$, so $y = 1$ and $z = 7$.

An important feature here is that lines parallel to the z-axis sometimes meet the plane $z = 6 + y$ first, and then $z = 8 - y$ later. As well as vice-versa. Things change when $y = 1$.

The xy projection is given in the problem; one readily finds that the slant line of that projection has equation $x + y = 4$ in the xy-plane, which also means that the vertical plane $x + y = 4$ in R^3 gives one of the sides of the region.

So we have the picture



Using the xy-projection, we obtain, for example

$$\int_0^1 \int_0^{4-y} \int_{6+y}^{8-y} f(x,y,z)\ dz\ dx\ dy + \int_1^4 \int_0^{4-y} \int_{8-y}^{6+y} f(x,y,z)\ dz\ dx\ dy.$$

Using the yz-projection, we obtain, for example

$$\int_0^1 \int_{6+y}^{8-y} \int_0^{4-y} f(x,y,z)\ dx\ dz\ dy + \int_1^4 \int_{8-y}^{6+y} \int_0^{4-y} f(x,y,z)\ dx\ dz\ dy.$$

Analyzing via an xz-projection is much more complicated since there are three nontrivial planes that meet lines parallel to the y-axis.

This is a more complicated variant of homework problem 52 in section 16.2; much of the wording is borrowed from that problem.

Problem 4) (20 Points)
4a) Let the oriented curve $\mathcal{C}$ be the first quadrant (i.e. $x \geq 0$ and $y \geq 0$) part of the circle $x^2 + y^2 = 4$ oriented so as to start at the point $(0,2)$ and end at the point $(2,0)$.

Find the value of the line integral

$$\int_{\mathcal{C}} -y\ dx\ +\ x\ dy.$$

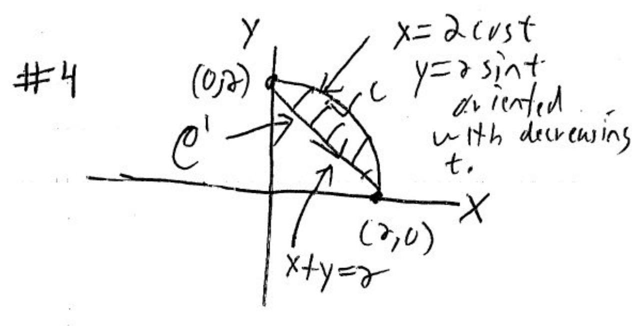
4b) Let $\mathcal{C}'$ be the straight line starting at $(0, 2)$ and ending at $(2, 0)$ with orientation in this direction.

Find the value of the line integral

$$\int_{\mathcal{C}'} -y \; dx \;+\; x \; dy.$$

4c) Use your results in parts a) and b), together with Green's theorem to find the area of the region between the curves $\mathcal{C}$ and $\mathcal{C}'$.

Solution:



4a) Thinking about polar coordinates, we see that $\mathcal{C}$ with the **opposite** to the requested parametrization may be described by

$$\begin{aligned} x &= \; 2\cos t \\ y &= \; 2\sin t \end{aligned}$$

where $0 \leq t \leq \frac{\pi}{2}$.

So we have

$$\begin{aligned} dx &= \; -2\sin t \; dt \\ dy &= \; 2\cos t \; dt \end{aligned}$$

and

$$\int_{\mathcal{C}} -y \; dx + x \; dy = -\int_0^{\frac{\pi}{2}} (-2\sin t)(-2\sin t) + (2\cos t)(2\cos t) \; dt = -4(\frac{\pi}{2}) = -2\pi.$$

4b) This line is $x + y = 2$ for $0 \leq x \leq 2$, so can be parametrized (with the as requested t increasing orientation) as

$$\begin{aligned} x &= \quad t \\ y &= \; 2 - t \end{aligned}$$

where $0 \leq t \leq 2$.

So we have

$$\begin{aligned} dx &= \quad dt \\ dy &= \; - \, dt \end{aligned}$$

and

$$\int_{\mathcal{C}'} -y \; dx \; + \; x \; dy = \int_0^2 -(2-t) + t(-1) \; dt = -4.$$

4c) By Green's theorem and the picture above of the region $\mathcal{R}$ (note $\mathcal{C}$ has the wrong orientation for the boundary of $\mathcal{R}$) between the two curves

$$\begin{aligned} \iint_{\mathcal{R}} \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) \;\; dx \; dy &= \qquad 2\text{Area}(\mathcal{R}) \\ &= \int_{\mathcal{C}'} -y \; dx \; + \; x \; dy - \int_{\mathcal{C}} -y \; dx \; + \; x \; dy \\ &= -4 - (-2\pi) = 2(\pi - 2). \end{aligned}$$

So Area$(\mathcal{R}) = \pi - 2$ as geometry also readily confirms.

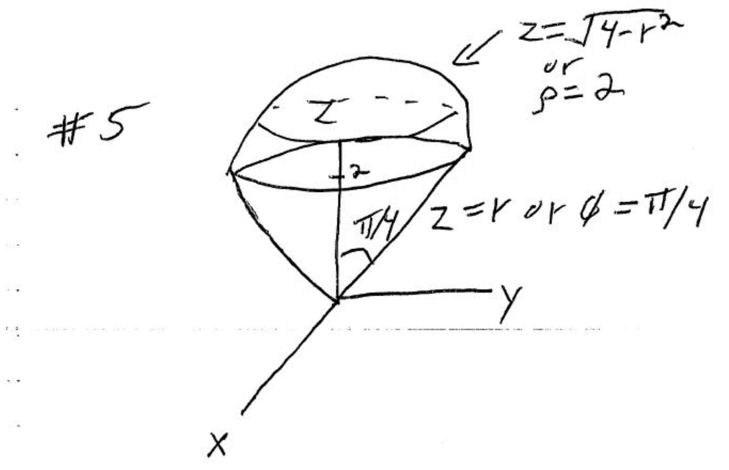
Problem 5) (20 Points) Consider the region above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere of radius 2 centered at the origin. (We are again viewing the postive z axis as the "up" direction.)

Let $f(x, y, z)$ be a differentiable function within this region (though not necessarily outside the region.)

5a) Set up in cylindrical coordinates (r, θ, z) the triple integral of $f(x, y, z)$ over this region. Include all limits of integration but do not attempt to compute the value of the integral.

5b) Set up in spherical coordinates (ρ, θ, ϕ) the triple integral of $f(x, y, z)$ over this region. Include all limits of integration but do not attempt to compute the value of the integral.

Solution:



5a) The cone and sphere meet when $x^2 + y^2 = z^2$ as well as $x^2 + y^2 + z^2 = 4$, so a circle in the $(z \geq 0)$ plane where $2z^2 = 4$, and $z = \sqrt{2} = r$. Hence using the picture above we have

$$\int_0^{\sqrt{2}} \int_0^{2\pi} \int_r^{\sqrt{4-r^2}} f(r\cos\theta, r\sin\theta, z)\ r\ dz\ d\theta\ dr.$$

5b) The cone is just given by $\phi = \frac{\pi}{4}$ (since a line of slope 1 through the origin makes a 45∘ angle with the vertical axis.) And the sphere is just $\rho = 2$. So we have

$$\int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_0^{2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\ \rho^2\sin\phi\ d\rho\ d\theta\ d\phi.$$

This is essentially homework problem 27 in section 16.5.